

# Uniqueness and Stability in Inverse Spectral Problems for Collapsing Manifolds

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Version September 25, 2012

**Abstract:** *We consider a geometric inverse problems associated with interior measurements: Assume that on a closed Riemannian manifold  $(M, h)$  we can make measurements of the point values of the heat kernel on some open subset  $U \subset M$ . Can these measurements be used to determine the whole manifold  $M$  and metric  $h$  on it? In this paper we analyze the stability of this reconstruction in a class of  $n$ -dimensional manifolds which may collapse to lower dimensions. In the Euclidean space, stability results for inverse problems for partial differential operators need considerations of operators with non-smooth coefficients. Indeed, operators with smooth coefficients can approximate those with non-smooth ones. For geometric inverse problems, we can encounter a similar phenomenon: to understand stability of the solution of inverse problems for smooth manifolds, we should study the question of uniqueness for the limiting non-smooth case. Moreover, it is well-known, that a sequence of smooth  $n$ -dimensional manifolds can collapse to a non-smooth space of lower dimension. To analyze the stability of inverse problem in a class of smooth manifolds with bounded sectional curvature and diameter, we study properties of the spaces which occur as limits of smooth collapsing manifolds and study the uniqueness of the inverse problems on the class of the limit spaces. Combining these, we obtain stability results for inverse problems in the class of smooth manifolds with bounded sectional curvature and diameter.*

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Inverse problem on a closed manifold . . . . .	4
1.2	Inverse problem for orbifolds . . . . .	10
1.3	Plan of the exposition . . . . .	14
<b>I</b>	<b>General case</b>	<b>14</b>
<b>2</b>	<b>Basic results in the theory of collapsing</b>	<b>15</b>
2.1	Basic properties of the limit spaces . . . . .	15
2.2	Fiber bundle theorems . . . . .	21
2.3	Measured Gromov-Hausdorff distance . . . . .	23
<b>3</b>	<b>Smoothness of the density functions</b>	<b>27</b>
<b>4</b>	<b>On properties of eigenfunctions</b>	<b>33</b>
<b>5</b>	<b>Continuity of the direct map</b>	<b>44</b>
5.1	Spectral estimates on $\overline{\mathfrak{M}\mathfrak{M}}$ and $\overline{\mathfrak{F}\mathfrak{M}\mathfrak{M}}$ . . . . .	46
5.2	Spectral convergence on $\overline{\mathfrak{F}\mathfrak{M}\mathfrak{M}}$ and $\overline{\mathfrak{M}\mathfrak{M}}$ . . . . .	54
5.3	Heat kernel convergence . . . . .	62
<b>6</b>	<b>From the local spectral data to the metric-measure structure</b>	<b>66</b>
6.1	Blagovestchenskii identity on $\overline{\mathfrak{M}\mathfrak{M}}_p$ . . . . .	68
6.2	Approximate controllability . . . . .	70
6.3	Cut locus . . . . .	74
<b>7</b>	<b>Stability of inverse problem</b>	<b>80</b>
<b>II</b>	<b>Orbifold case</b>	<b>87</b>

<b>8</b>	<b>Volume growth and dimensionality of collapse</b>	<b>87</b>
<b>9</b>	<b>From isometry to isomorphism</b>	<b>91</b>
9.1	Good orbifolds . . . . .	93
9.2	Ball-to-ball continuation: center point . . . . .	95
9.3	Ball-to-ball continuation . . . . .	99
9.4	Continuation along balls. . . . .	103
9.5	Completing proof of Theorem 9.3 . . . . .	107
9.6	General Case . . . . .	108
<b>10</b>	<b>Appendix A: Collapsing manifolds in physics</b>	<b>109</b>
<b>11</b>	<b>Appendix B: Remark on the smoothness in Calabi-Hartman [12] and Montgomery-Zippin [47]</b>	<b>114</b>
<b>12</b>	<b>Appendix C: Operator-theoretical approach to operator <math>\Delta_X</math>.</b>	<b>123</b>

# 1 Introduction

In classical inverse problems one wants to show that physical measurements can be used to determine coefficients of various partial differential equations modeling macroscopic and microscopic phenomena. Examples of these are the paradigm problems, the inverse problem for conductivity equation or inverse scattering problem for Schrödinger operator [4, 13, 39, 48, 49, 50, 59]. In these problems the structure of the underlying (Euclidean) space is *a priori* known before measurements. Recently many inverse problems have been generalized to invariant settings, for cases where the underlying space is not a priori known but is assumed to be a Riemannian manifold. The given measurements can be either measurements on the boundary of this manifold (if the boundary is non-empty) or in an interior subdomain of the manifold.

In this paper we consider the inverse problems associated with interior measurements: Assume that on a closed Riemannian manifold  $(M, h)$  we can make measurements on some open subset  $U \subset M$ . Can these measurements be used to determine the whole manifold  $M$  and metric  $h$  on it? In this pa-

per we analyse the stability of this reconstruction in a class of  $n$ -dimensional manifolds which may collapse to lower dimensions. In Euclidian space, stability results for inverse problems for partial differential operators need considerations of operators with non-smooth coefficients. Indeed, operators with smooth coefficients can approximate those with non-smooth ones. Moreover, the study of inverse problems in the Euclidian space has led, in the case of very non-smooth metrics, to the counterexamples where even the change of the topology of the domain is not observed in the measurements [32]. These, in turn, have led to some engineering applications in the emerging field of transformation optics and invisibility cloaking [33, 34, 35]. From the point of view of stability, it is clear that, approaching an invisibility limit, we can not expect any stability of the inverse problem for the smooth case. For geometric inverse problems, we can encounter a similar phenomenon: to understand stability of the solution of inverse problems for smooth manifolds, we should study the question of uniqueness for the limiting non-smooth case. Moreover, it is well-known, see e.g. [36], that a sequence of smooth  $n$ -dimensional manifolds can collapse to a non-smooth space of lower dimension. Thus, a similar phenomenon of invisibility can occur in geometric inverse problems. Therefore, an extra care should be taken to formulate conditions on a class of Riemannian manifolds which, on one hand, would allow for substantial generality, i.e. for a collapse to lower dimensions, and, on the other hand, would avoid invisible limits. In particular, to analyse stability in a desired class of smooth manifolds, one should study uniqueness of inverse problems in the spaces which occur as limits of these collapsed manifolds. Also, as explained in Appendix A, the study of inverse problems on collapsed manifolds is encountered in various models of the modern physics.

## 1.1 Inverse problem on a closed manifold

Let  $(M, h)$  be a closed, compact, connected Riemannian manifold without boundary. First, we assume that the manifold has  $C^\infty$ -smooth coordinates and  $C^\infty$ -smooth metric tensor,  $h$ . We denote by  $dV = dV_h$  its Riemannian volume, and by  $d\mu_M$  its normalised probability measure.

$$d\mu_M = \frac{dV_h}{\text{Vol}(M)}, \quad (1)$$

where  $\text{Vol}(M)$  is the Riemannian volume of  $(M, h)$ . In the future, we deal with pointed Riemannian manifolds  $(M, p)$  and typically skip in our notations

indications of the metric tensor  $h$ , denoting the Riemannian manifold by  $(M, p, \mu_M)$  or often  $(M, p)$ , or just  $M$ .

Let  $\Delta_M$  be the Laplacian on  $(M, h)$  which, in local coordinates on  $M$ , takes the form

$$\Delta_M u := |h|^{-1/2} \partial_j (|h|^{1/2} h^{jk} \partial_k u), \quad |h| = \det(h_{jk}), \quad j, k = 1, \dots, n. \quad (2)$$

Consider the heat kernel,  $H(x, y, t) = H(x, y, t)$ , associated to this operator,

$$(\partial_t - \Delta_M)H(\cdot, y, t) = 0 \quad \text{on } M \times \mathbb{R}_+, \quad H(\cdot, y, 0) = \delta_y, \quad (3)$$

where  $\delta_y$  is the normalized Dirac delta-distribution,  $\int_M \delta_y(x) \phi(x) d\mu_M(x) = \phi(y)$  for  $\phi \in C^\infty(M)$ .

In the following we assume that we are given the values of the heat kernel at points  $\{z_\alpha : \alpha = 1, 2, \dots\}$  which form a dense set in a ball  $B = B_M(p, r)$  of  $(M, p)$ , having center at  $p$  and radius  $r > 0$ . To this end, we define pointwise heat data  $PHD = PHD(M, p)$  to be the collection of the following ordered sequences

$$PHD = \{H_{\alpha, \beta, \ell}\}_{\alpha, \beta, \ell=1}^\infty, \quad H_{\alpha, \beta, \ell} = H(z_\alpha, z_\beta, t_\ell), \quad (4)$$

where  $(t_\ell)_{\ell=1}^\infty$  is a dense set of  $\mathbb{R}_+ = (0, \infty)$ . Let us emphasize that the mutual relations of the measurement points  $z_\alpha$ , e.g. the distances between these points, their position with regard to  $p$ , as well as  $p$  itself, are not *a priori* known.

Let  $(M, p)$  and  $(M', p')$  be two smooth Riemannian manifolds. We say that their PHD,  $\{H_{\alpha, \beta, \ell}\}_{\alpha, \beta, \ell=1}^\infty, \{H'_{\alpha, \beta, \ell}\}_{\alpha, \beta, \ell=1}^\infty$  are *equivalent* if

$$H_{\alpha, \beta, \ell} = H'_{\alpha, \beta, \ell}, \quad \text{for all } \alpha, \beta, \ell = 1, \dots \quad (5)$$

We consider the following generalization of the Gel'fand inverse problem, [29]

**Problem 1.1** *Let PHD of two smooth, compact, connected Riemannian manifolds  $(M, p)$  and  $(M', p')$  coincide, i.e. satisfy (5). Are these manifolds isometric?*

The positive answer to this question is given in [40, 41, 44], see also. Our plan is to analyse the stability of the solution of this problem using the Boundary Control method, see e.g. [5, 46, 41].

We work in the class  $\mathfrak{MM}_p = \mathfrak{MM}_p(\Lambda, D, n)$  of pointed closed Riemannian manifolds  $(M, p, \mu_M)$ , satisfying

$$|R(M)| \leq \Lambda^2, \quad \text{diam}(M) \leq D, \quad \dim(M) = n, \quad (6)$$

where  $R(M)$  is the sectional curvature on  $M$ . For example, all flat tori  $S^1 \times S_\varepsilon^1$ , where  $S_\varepsilon^1$  is a circle of radius  $\varepsilon$  and  $S^1 = S_1^1$ , endowed with their standard metric, satisfy the above conditions. In the following, we denote by  $d_M(x, y)$  the distance between  $x$  and  $y$  on  $M$  and say that a set  $A \subset M$  is  $\delta$ -dense in  $B \subset M$ , or a  $\delta$ -net in  $B$  if, for any  $y \in B$ , there is  $x \in A$  such that  $d_M(x, y) < \delta$ . Our aim is to prove the following stability theorem for the solution of inverse problem (1.1).

**Theorem 1.2** *Let  $r, \Lambda, D > 0$ ,  $n \in \mathbb{Z}_+$ . Let  $\mathfrak{MM}_p$  be a class of pointed Riemannian manifolds  $(M, p, \mu_M)$  defined by condition (6), i.e. having dimension  $n$  with sectional curvature bounded by  $\Lambda$  (from above and below) and diameter bounded by  $D$ .*

*Then, there exists an increasing function  $\omega(s) = \omega_{(r, \Lambda, D, n)}(s)$ ,*

$$\omega : [0, 1) \rightarrow [0, \infty), \quad \lim_{s \rightarrow 0} \omega(s) = 0, \quad (7)$$

*with the following properties:*

*Assume that  $(M, p, \mu_M), (M', p', \mu_{M'}) \in \mathfrak{MM}_p$ . Let  $\{z_\alpha\}_{\alpha=1}^N \subset M$ ,  $\{z'_\alpha\}_{\alpha=1}^N \subset M'$  be  $\delta$ -nets in  $B(p, r)$ ,  $B'(p', r)$ , correspondingly, and  $\{t_l\}_{l=1}^L$  be a  $\delta$ -net in  $(\delta, \delta^{-1})$ , where  $N = N(\delta)$ ,  $L = L(\delta)$ . Assume also that*

$$|H(z_\alpha, z_\beta, t_l) - H'(z'_\alpha, z'_\beta, t_l)| < \delta, \quad 1 \leq \alpha, \beta \leq N, \quad 1 \leq l \leq L, \quad (8)$$

*where  $H, H'$  are the heat kernels on  $M, M'$ , correspondingly.*

*Then*

$$d_{pmGH}((M, p, \mu_M), (M', p', \mu_{M'})) < \omega(\delta). \quad (9)$$

Here  $d_{pmGH}(M, M')$  stands for the *pointed measured Gromov-Hausdorff distance* between (compact) pointed metric-measure spaces. Namely, we say that  $d_{pmGH}(M, M') < \varepsilon$ , if there are measurable maps  $\psi : M \rightarrow M'$  and  $\psi' : M' \rightarrow M$  such that,

(i) for any  $x, y \in M$ ,  $x', y' \in M'$ ,

$$\begin{aligned} |d_{M'}(\psi(x), \psi(y)) - d_M(x, y)| &< \varepsilon, \quad d_{M'}(\psi(p), p') < \varepsilon, \\ |d_M(\psi'(x'), \psi'(y')) - d_{M'}(x', y')| &< \varepsilon, \quad d_M(\psi'(p'), p) < \varepsilon. \end{aligned} \quad (10)$$

(ii) for all Borel sets  $A \subset M$  and  $A' \subset M'$ ,

$$\mu(\psi^{-1}(A')) < \mu'((A')^\varepsilon) + \varepsilon, \quad \mu'((\psi')^{-1}(A)) < \mu(A^\varepsilon) + \varepsilon, \quad (11)$$

where  $A^\varepsilon$  denotes the  $\varepsilon$ -neighborhood of  $A$ ;  $A^\varepsilon := \{x \in M : d_M(x, A) < \varepsilon\}$  and similar for  $(A')^\varepsilon$ , and we abbreviate  $\mu = \mu_m$ ,  $\mu' = \mu_{M'}$ .

As measuring the values of physical quantities at a point may be difficult, we make the following remark on the result of Theorem 1.2:

**Remark 1.3** The point values of the heat kernels in (8) of Theorem 1.2 can be replaced by the condition

$$|\tilde{H}_{\alpha\beta\ell} - \tilde{H}'_{\alpha\beta\ell}| < \delta \quad \text{for all } 1 \leq \alpha, \beta \leq N, \ 1 \leq \ell \leq L. \quad (12)$$

Here, for the manifold  $(M, p, \mu)$ ,

$$\begin{aligned} \tilde{H}_{\alpha\beta\ell} &= \frac{1}{\mu_\alpha \mu_\beta} \int_{B_M(z_\alpha, \varepsilon) \times B_M(z_\beta, \varepsilon)} H(x, x', t_\ell) d\mu_M(x) d\mu_M(x'), \\ \mu_\alpha &= \mu_M(B_M(z_\alpha, \varepsilon)), \quad \mu_\beta = \mu_M(B_M(z_\beta, \varepsilon)) \end{aligned}$$

is the heat kernel averaged over small balls, and  $\tilde{H}'_{\alpha\beta\ell}$  are defined similarly on  $(M', p', \mu')$ . From the physical point of view this means that, instead of the point values the heat kernel, it is enough to observe the total amount of heat inside small balls near the measurement points  $z_\alpha$ .

The proof of Theorem 1.2 is based on considering the inverse problem on  $\overline{\mathfrak{MM}}_p$ . This is the closure, in the pointed measured GH topology, of the moduli space  $\mathfrak{MM}_p$  defined by (6). The structure of the pointed metric-measure spaces  $(X, p) \in \overline{\mathfrak{MM}}_p$  was studied in [23]–[25]. It was shown that  $\overline{\mathfrak{MM}}_p$  consists of stratified Riemannian manifolds of dimension  $d = n - k$ ,  $k \geq 0$ , where  $k$  is called the dimension of collapse. Note that  $X = X^{reg} \cup X^{sing}$ , where the regular part,  $X^{reg}$  is a  $d$ -dimensional Riemannian manifold with  $C_*^2$ -metric  $h_X$  (for the smoothness of  $h_X$ , as well as the density function  $\rho_X$

introduced later, see [24] and also [38] and section 3 and Appendix B below.) The space  $X$  is equipped with a probability measure

$$d\mu_X = \rho_X \frac{dV}{\text{Vol}(X)}, \quad \mu_X(X^{sing}) = 0,$$

where  $\rho_X > 0$  is  $C_*^2$ -smooth. An elliptic operator, naturally associated with  $(X, p, \mu_X)$  is the *weighted Laplacian*,  $\Delta_X$ . In local coordinates on  $X^{reg}$ ,

$$\Delta_X u = \frac{1}{\rho_X |h_X|^{1/2}} \frac{\partial}{\partial x^j} \left( \rho_X |h_X|^{1/2} h_X^{jk} \frac{\partial}{\partial x^k} u \right),$$

cf.(2). It was shown in [24] with further ramifications in [38] and section 5 and Appendix C, that the pointed measured GH-convergence in  $\overline{\mathfrak{MM}}_p$  implies the spectral convergence. In particular, see Theorem 5.17 below, if  $d_{pmGH}(M_\ell, X) \rightarrow 0$ , for  $M_\ell \in \mathfrak{MM}_p$ , then, the heat kernels  $H_\ell$  on  $M_\ell$  converge, in a proper sense, to the heat kernel  $H$  on  $X$ , i.e. the "direct" spectral problem on  $\overline{\mathfrak{MM}}_p$  is continuous with respect to the pointed measured GH convergence. Moreover, it is shown in section 6 that the local PHD uniquely determine the metric-measure structure of  $X$ . Then, the stability Theorem 1.2 follows from this uniqueness and the continuity of the direct problem by the standard compactness-type arguments, see section 7.

We illustrate the nature of the pointed measured GH convergence and its relation to the spectral convergence by the following example:

**Example 1.4** [24] Let  $(M^\sigma = S^1 \times S^1, p^\sigma)$  be a torus, identified with the square  $[-1, 1]^2$  with boundaries glued together and local coordinates  $(y, z) \in (-1, 1)^2$  with  $p^\sigma = O$ . The warped product-type metric tensor  $h^\sigma$  on  $M^\sigma$  is defined by

$$ds^2 = dy^2 + \sigma^2 c^2(y) dz^2, \quad c(y) > 0. \quad (13)$$

Consider the eigenvalues,  $\lambda_j^\sigma$ , counting multiplicity, and the corresponding normalised eigenfunctions,  $\phi_j^\sigma(y, z)$ , of the Laplacians,  $\Delta_\sigma := \Delta_{M^\sigma}$ ,

$$(\Delta_\sigma + \lambda_j^\sigma) \phi_j^\sigma(y, z) = 0, \quad \langle \phi_j^\sigma, \phi_k^\sigma \rangle_{L^2(M^\sigma, \mu^\sigma)} = \delta_{jk}, \quad \mu^\sigma = \mu_{M^\sigma}.$$

Due to (13), for each  $j$  there are  $m = m(j) \in \mathbb{Z}$  and  $\ell = \ell(j) \in \mathbb{Z}_+$  such that

$$\phi_j^\sigma(y, z) = e^{imz} \Phi_{\ell, m}^\sigma(y),$$



where  $\Phi_{\ell,m}^\sigma(y)$  satisfy

$$-\frac{1}{c(y)}\frac{\partial}{\partial y}(c(y)\frac{\partial}{\partial y}\Phi_{\ell,m}^\sigma(y)) + \frac{m^2}{c(y)^2\sigma^2}\Phi_{\ell,m}^\sigma(y) = \lambda_{\ell,m}^\sigma\Phi_{\ell,m}^\sigma(y),$$

$$\langle \Phi_{\ell,m}^\sigma, \Phi_{\ell',m}^\sigma \rangle_{L^2(S,\mu_S)} = \delta_{\ell,\ell'},$$

with  $S = S^1$ ,  $d\mu_S = c(y)dy$  and  $\lambda_j^\sigma = \lambda_{\ell,m}^\sigma$ . For  $m = 0$  the eigenvalues  $\lambda_{\ell,0}^\sigma =: \widehat{\lambda}_\ell$  are independent of  $\sigma$  and the corresponding eigenfunctions  $\Phi_{\ell,0}^\sigma(y, z) =: \widehat{\Phi}_\ell(y)$  are independent of  $\sigma$  and  $z$ . For  $m \neq 0$

$$\lambda_{\ell,m}^\sigma \geq \frac{m^2}{\sigma^2} \min_{y \in S^1} c^{-2}(y).$$

Thus, as  $\sigma \rightarrow 0$ , all the eigenvalues corresponding to  $m \neq 0$  tend to  $\infty$ . By re-ordering for each  $\sigma$  the eigenvalues  $\lambda_j^\sigma$ ,  $j = 1, 2, \dots$ , in the increasing order, we see that, for any  $j = 1, \dots$ ,  $\lambda_j^\sigma \rightarrow \widehat{\lambda}_j$  and the corresponding eigenfunctions  $\phi_j^\sigma(y, z)$  converge pointwise to  $\widehat{\Phi}_j(y)$ .

Let  $H_\sigma((y, z), (y', z'), t)$  be the heat kernels of  $(M^\sigma, \mu^\sigma)$  and  $H(y, y', t)$  be the heat kernel of  $(S, \mu_S)$ . Then,

$$\lim_{\sigma \rightarrow 0} H^\sigma((y, z), (y', z'), t) = H(y, y', t)$$

for all  $(y, z), (y', z') \in M$  and  $t > 0$ .

Next, for any  $n \in \mathbb{Z}_+$ , we choose  $N = N(n)$  points of form  $(y_\alpha, z_\alpha^n)$ ,  $\alpha = 1, \dots, N$ , which form a  $(1/n)$ -net in  $M^{1/n}$ , while  $t_\ell$ ,  $\ell = 1, \dots, L$ ,  $L = L(n)$ , form a  $1/n$ -net on  $(1/n, n)$ . Then PHD of  $(M^{1/n}, \mu^{1/n})$  tend, as  $n \rightarrow \infty$ , to PHD of  $(S, \mu_S)$  in the sense, that

$$\lim_{n \rightarrow \infty} |H_{1/n}((y_\alpha, z_\alpha^n), (y_\beta, z_\beta^n), t_\ell) - H(y_\alpha, y_\beta, t_\ell)| = 0$$

uniformly with respect to  $\alpha, \beta \in \{1, \dots, N\}$ ,  $\ell = \{1, \dots, L\}$ .

Note that Theorem 1.2 is of a global nature providing the uniform rate of the metric-measure convergence on  $\mathfrak{MM}_p$ . When dealing with a local convergence, in a neighborhood of a given manifold  $M \in \mathfrak{MM}_p$ , it is possible to obtain more precise information about the structure of spaces which are close to  $M$ . Namely, we have the following result

**Corollary 1.5** *Let  $(M, p, \mu_M) \in \mathfrak{MM}_p$ . There is  $\delta_M > 0$  such that, if  $(M', p', \mu_{M'}) \in \mathfrak{MM}_p$  satisfies conditions of Theorem 1.2 with  $\delta < \delta_M$  in (8), then  $M'$  is  $C^3_*$ -diffeomorphic to  $M$ .*

*Moreover, taking this diffeomorphism between  $M$  and  $M'$  to be identity, for any  $\alpha < 1$ ,*

$$\|h_{M'} - h_M\|_{C^{1,\alpha}} \leq \omega^\alpha(\delta), \quad \text{where } \omega^\alpha(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (14)$$

Note that to compare, in  $C^{1,\alpha}(M)$ , the metric tensors  $h_M$  and  $h_{M'}$  we should use proper, e.g. harmonic, coordinates [1], [2].

## 1.2 Inverse problem for orbifolds

Corollary 1.5 shows that it is sometimes possible to obtain more information about the nature of the convergence than given by Theorem 1.2. In particular, if we know *a priori* that the dimension of collapse is at most 1, then the limiting space  $X$  is, at worst, an orbifold. This result follows implicitly from [26], see also Corollary 2.4.

Let us recall the basic definitions regarding Riemannian orbifolds, see e.g. [63], Ch. 13 and also [55], where the notion of the orbifold, called  $V$ -manifold was introduced, [17], where the Riemannian structure on orbifolds was considered and [20] for further study of the spectral properties of its Laplacian. We say that a topological (metric) space  $(X, p)$  is a (Riemannian) orbifold of dimension  $d$ , if any point  $x \in X$  has a neighborhood  $U$  such that

(i) there is a map  $\pi : \tilde{U} \rightarrow U$ , where  $\tilde{U} = B(r) \subset \mathbb{R}^d$  is a ball, with respect to some metric  $\tilde{h}$ , of radius  $r > 0$  centered at origin  $O$ ;

(ii) there is a finite group  $G(x) \subset O(d)$  acting on  $\tilde{U}$ , such that

$$g^*(\tilde{h}) = \tilde{h}, \quad \pi(O) = x, \quad \pi(g \circ x') = \pi(x') \text{ for } x' \in \tilde{U}, \quad g \in G(x); \quad (15)$$

(iii) the metric  $h$  and  $\tilde{h}$  are related by

$$\tilde{h} = \pi^*(h). \quad (16)$$

Note that the Riemannian metric tensor  $h$  on  $X$  is defined on  $X^{reg}$ , where  $X^{reg} = \{x \in X : G(x) = Id\}$ .

In this case, we say that  $\pi : \tilde{U} \rightarrow U$  is a uniformising cover at  $x$ . Later in the text we refer to  $U$  as a coordinate chart and  $(x_1, \dots, x_d) \in \tilde{U}$  as coordinates near  $x$ .

We can consider  $X$  as a metric space by endowing it with a distance function  $d_X$  such that, for any  $x \in X$  and uniformising cover  $\pi : \tilde{U} \rightarrow U$  at  $x$ , if  $y_1, y_2 \in \pi^{-1}(B(x, r))$  with small  $r$ , then

$$d_X(y_1, y_2) = \min_{\tilde{y}_i \in \pi^{-1}(y_i), i=1,2} d_{\tilde{U}}(\tilde{y}_1, \tilde{y}_2).$$

Then  $X$  is a locally compact and complete length space, and thus is a geodesic space, that is, all its points can be connected by a length minimizing curve.

As explained in subsection 1.1, when considering the pointed measured GH-convergence in  $\mathfrak{MM}_p$  which gives rise to a 1-dimensional collapse, the resulting Riemannian orbifolds are endowed with a  $C_*^2$ -smooth density function  $\rho_X$ . Note that in this case the density function can be lifted to uniformising cover. Namely, for a coordinate chart  $U$  at  $x \in X$ , there exists a density function  $\tilde{\rho}_{\tilde{U}}$  on  $\tilde{U}$  so that

$$\tilde{\rho}_{\tilde{U}} = \pi^*(\rho_U), \quad g^*(\tilde{\rho}_{\tilde{U}}) = \tilde{\rho}_{\tilde{U}}, \text{ for } g \in G(x).$$

Due to the extra group structure at singular points, when speaking about Riemannian orbifolds with measure, one can introduce the notion of *isomorphism of orbifolds* going back to [55]. Namely, two orbifolds  $(X, p, \mu)$  and  $(X', p', \mu')$  are isomorphic with isomorphism  $\Phi$ , if  $\Phi : X \rightarrow X'$  is a homeomorphism with the following properties:

1.  $\Phi(p) = p'$ ;
2. for any  $x \in X$  and  $x' = \Phi(x) \in X'$  there are neighborhoods  $U$  and  $U'$  with a uniformizing cover  $\pi : \tilde{U} \rightarrow U$  and a metric  $\tilde{h}$  on  $\tilde{U}$  and a uniformizing cover  $\pi' : \tilde{U}' \rightarrow U'$  and a metric  $\tilde{h}'$  on  $\tilde{U}'$ , such that
  - (i)  $\Phi(U) = U'$ ;
  - (ii) there exists a Riemannian isometry  $\tilde{\Phi} : \tilde{U} \rightarrow \tilde{U}'$ ;

- (iii) there exists an isomorphism  $\widehat{\Phi}_x : G(x) \rightarrow G'(x')$ , where the group  $G(x) \subset O(d)$  corresponds to  $x$ , and the group  $G'(x') \subset O(d)$  corresponds to  $x'$ ;
- (iv) the following equations hold true:

$$\begin{aligned} (\widetilde{\Phi})^* \widetilde{h}' &= \widetilde{h}, \quad (\widetilde{\Phi})^* \widetilde{\rho}' = \widetilde{\rho}; \\ \Phi \circ \pi &= \pi' \circ \widetilde{\Phi} \text{ on } \widetilde{U}, \quad \widetilde{\Phi}(g \circ \widetilde{x}) = \widehat{\Phi}(g) \circ \widetilde{\Phi}(\widetilde{x}) \text{ for } \widetilde{x} \in \widetilde{U}. \end{aligned} \tag{17}$$

Clearly that, similar to the manifold case, we can define PHD for  $C_*^2$ -smooth Riemannian orbifolds with measure. Here  $C_*^2$ -smoothness means that, in proper coordinates on any uniformising chart  $\widetilde{U}$ , the metric tensor  $\widetilde{h}$  and the density function  $\widetilde{\rho}$  are  $C_*^2$ -smooth on  $\widetilde{U} \cap \pi^{-1}(X^{reg})$ . Then we have the following generalization of the Gel'fand inverse problem:

**Problem 1.6** *Let PHD of two  $C_*^2$ -smooth, compact, connected Riemannian orbifolds with measure  $(X, p, \mu)$  and  $(X', p', \mu')$  coincide, i.e. satisfy (5). Are these orbifolds isomorphic?*

Note that, since any Riemannian manifold equipped with its Riemannian probability measure is an orbifold with  $\rho = 1$  Problem 1.6 is a generalization of Problem 1.1.

**Theorem 1.7** *Let  $(X, p, \mu)$ ,  $\partial X = \emptyset$ , and  $(X', p', \mu')$ ,  $\partial X' = \emptyset$ , be two  $C_*^2$ -smooth Riemannian orbifolds with measure. Then, if PHD for  $X$  is equivalent to PHD for  $X'$ , in the sense of (5), then  $(X, p, \mu)$  and  $(X', p', \mu')$  are isometric (as metric spaces) with an isometry  $\Phi : X \rightarrow X'$ , such that  $\Phi(z_\alpha) = z'_\alpha$ ,  $\alpha = 1, \dots, M$ ,  $\Phi(p) = p'$  and  $\Phi^*(h') = h$ ,  $\Phi^*(\rho') = \rho$ .*

*Moreover,  $(X, p, \mu)$  and  $(X', p', \mu')$  are isomorphic with  $\Phi$  being an isomorphism.*

**Remark 1.8** We formulate Theorem 1.7 only for  $C_*^2$ -smooth orbifolds without boundary. However, the result remains valid if we assume that  $h, h'$  are  $C_*^2$ -smooth while  $\rho, \rho'$  are  $C^{0,1}$ -smooth.

To apply Theorem 1.7 to study stability of inverse problems on smooth manifolds, we should formulate the conditions which guarantees that the collapse is, at most, 1-dimensional and gives rise to an orientable Riemannian

orbifold. To analyse the dimensionality of collapse, we introduce the class of pointed Riemannian manifolds  $(M, p)$ ,  $\mathfrak{M}_k = \mathfrak{M}_{k,p}(n, \Lambda, D; c_0)$ , where  $\Lambda, D > 0$ ,  $n \in \mathbb{Z}_+$ ,  $k \in \mathbb{Z}_+ \cup \{0\}$ ,  $k \leq n$ . The class  $\mathfrak{M}_{k,p}$  is identified by condition (6) and

$$\mu_M(B_h(x, r)) \leq c_0 r^{n-k}, \quad \text{for some } x \in M, \quad 0 < r < c_0^{-1}. \quad (18)$$

Note that we use the notation  $\mathfrak{M}_{k,p}$  rather than  $\mathfrak{MM}_{k,p}$  when we are interested in the pointed GH-convergence defined by (10) rather than the pointed measured GH-convergence defined by (10), (11).

**Lemma 1.9** *Let a sequence  $(M_i, p_i)$  in  $\mathfrak{M}_{k,p}(n, \Lambda, D; c_0)$  GH-converge to a pointed space  $(X, p)$ . Then  $\dim X \geq n - k$ .*

Together with Corollaries 2.4 and 2.7, Theorem 1.7 and Lemma 1.9 give rise to the following result:

**Corollary 1.10** *There exists  $\delta_0 = \delta_0(r, \Lambda, D, n, c_0)$  such that, if  $(M, p, \mu_M)$  and  $(M', p', \mu_{M'})$  from  $\mathfrak{MM}_{n-1,p}$  satisfy (8) with  $\delta < \delta_0$ , then*

- i. *either  $M$  and  $M'$  are diffeomorphic,*
- ii. *or  $M$  and  $M'$  are  $S^1$ -Seifert fibrebundles on isomorphic orbifolds  $X$  and  $X'$ .*

We complete this subsection with an example illustrating collapse to an orbifold and structure of the singular points on it.

**Example 1.11** For  $\varepsilon > 0$  let  $M_\varepsilon$  be a 3-dimensional Riemannian manifold of the following form: We start with a cylinder  $\mathbb{S}^2 \times [0, \varepsilon]$ , where  $\mathbb{S}^2$  is the 2-dimensional unit sphere with canonical metric. Assuming  $\mathbb{S}^2 \subset \mathbb{R}^3$ , we define the action of the group  $Z_m$ ,  $m \in \mathbb{Z}_+$ , by  $2\pi/m$ -rotations around the  $z$ -axis. We then identify points  $(\mathbf{x}, 0) \in \mathbb{S}^2 \times \{0\}$  with  $(e^{(2\pi i/m)} \circ \mathbf{x}, \varepsilon) \in \mathbb{S}^2 \times \{\varepsilon\}$ , where  $e^{(2\pi i/m)} \circ \mathbf{x}$  stands for the rotation by  $2\pi/m$ , to obtain  $M_\varepsilon$ . Note that this gives rise to the closed vertical geodesics of the length  $m\varepsilon$  except for the points, corresponding to the north,  $N$ , and south,  $S$ , poles of  $\mathbb{S}^2$  which give rise to the closed vertical geodesics of length  $\varepsilon$ . When  $\varepsilon \rightarrow 0$ ,  $M_\varepsilon$  collapse to a 2-dimensional Riemannian orbifold  $X$  which has two singular conic points,  $N$  and  $S$  with  $G(N) = G(S) = Z_m$ .

### 1.3 Plan of the exposition

The paper consists of two parts plus three Appendices. Part I, Sec. 2–Sec. 7, deals with the proof of Theorem 1.2 and related topics. Sec. 2 is of the expository nature. Namely, we review, in a somehow modified form appropriate for our purposes, Fukaya’s results on the measured GH convergence of Riemannian manifolds and provide some further results in this direction. In Sec. 3 we show that, if  $X \in \overline{\mathfrak{MM}}_p$ , then its density function  $\rho \in C_*^2(X)$ . This improves on the earlier results in [24] and [38]. Note that our proof differs from that in the above papers as it is based on the analysis of smoothness of transformation groups into the scale of Zygmund-type function. In turn, this requires an extension of the classical Montgomery-Zippin results, [47], which is done in Appendix B. Sec. 4 is of an auxiliary nature. Here we prove various results concerning the behavior of spectrum, eigenfunctions and heat kernels on  $\overline{\mathfrak{MM}}_p$  and analyse the spectral information contained in PHD. In Sec. 5 we continue to study the spectral behavior on  $\overline{\mathfrak{MM}}_p$  obtaining some uniform estimates for the eigenfunctions and heat kernels and prove the spectral convergence of the corresponding weighted Laplacians with respect to the pointed measured GH-convergence. Some auxiliary results dealing with the relations between the Laplacians on  $M$  and its frame bundle  $TM$  as well as the corresponding structures on  $X \in \overline{\mathfrak{MM}}_p$  are considered in Appendix C. In Sec. 6, extending the geometric BC-method, see e.g. [41], to  $\overline{\mathfrak{MM}}_p$  we show that PHD of any  $X \in \overline{\mathfrak{MM}}_p$  uniquely determine its metric-measure structure. At last, Sec. 7 is devoted to the proof of Theorem 1.2.

Part II, Sec. 8 – Sec. 9 is related to the proof of Theorem 1.7. Namely, in Sec. 8 we study the relation between the volume growth condition (18) and dimensionality of collapse. At last, Sec. 9 explains how, in the case of an orientable Riemannian orbifold with measure, its metric-measure structure determines its isomorphy-type. Moreover, since our proof works also for the orbifolds with boundary, Theorem 9.1 deals with this more general case. Appendix A provides a brief overview on the use of collapsing manifolds in physics, in particular, that to the orbifolds.

## Part I

# General case

## 2 Basic results in the theory of collapsing

For given  $n \in \mathbb{Z}_+$  and  $\Lambda, D > 0$ ,  $\mathfrak{M}_p(n, \Lambda, D)$  stands for the class of  $n$ -dimensional closed pointed Riemannian manifolds  $M$  satisfying

$$|R(M)| \leq \Lambda^2, \text{ diam}(M) \leq D,$$

where  $R(M)$  and  $\text{diam}(M)$  stand for the sectional curvature and the diameter of the manifold  $(M, h)$ .

The structure of collapsing in the moduli space  $\mathfrak{M}_p(n, \Lambda, D)$  was extensively studied by Fukaya ([25], [24], [26]). In this section, we mainly review some of them reformulating them for the case of pointed manifolds.

### 2.1 Basic properties of the limit spaces

It is known (see [36]) that  $\mathfrak{M}_p(n, \Lambda, D)$  is precompact in the Gromov-Hausdorff distance, (10). This means that any sequence  $M_i \in \mathfrak{M}_p(n, \Lambda, D)$ ,  $i = 1, 2, \dots$ , contains a subsequence, which we can assume to coincide with the whole sequence, converging to some compact metric space  $X$  with respect to the Gromov-Hausdorff distance. We start with a characterization of  $X$ .

Fix any point  $q \in X$  and put  $q_i := \psi_i(p)$ , where  $\psi_i : X \rightarrow M_i$  is an  $\varepsilon_i$  Gromov-Hausdorff approximation ( $\varepsilon_i$  GH-approximation, or  $\varepsilon_i$ -approximation in short) with  $\lim \varepsilon_i = 0$ . Namely, it satisfies

$$\begin{aligned} |d_i(\psi(x), \psi(y)) - d_X(x, y)| &< \varepsilon_i, \\ \psi_i(X) &\text{ is } \varepsilon_i\text{-dense in } M_i, \end{aligned} \tag{19}$$

where  $d_i = d_{M_i}$ . Let  $B$  be the open ball around the origin  $O$  in  $\mathbb{R}^n$  of radius  $\pi/\Lambda$ , and let  $\exp_i : B \rightarrow M_i$  be the composition of  $\exp_{q_i} : T_{q_i}(M_i) \rightarrow M_i$  and a linear isometric embedding  $B \rightarrow B(O, \pi/\Lambda) \subset T_{q_i}(M_i)$ . Since  $|R(M_i)| \leq \Lambda^2$ ,  $\exp_i : B \rightarrow M_i$  has maximal rank. Thus, we have the pull-back metric  $\tilde{h}_i := \exp_i^*(h_{M_i})$  on  $B$ . Moreover, we see that the injectivity radius  $\text{inj}(B, \tilde{h}_i)$

is uniformly bounded from below. Therefore, we may assume that  $(B, \tilde{h}_i)$  converges to a  $C_*^2$ -metric  $(B, \tilde{h}_0)$  with respect to  $C^{1,\alpha}$ -topology, for any  $0 < \alpha < 1$  (see [31], [52], [1], [2]).

Let  $G_i$  denote the set of all isometric embeddings  $\gamma : (B', \tilde{h}_i) \rightarrow (B, \tilde{h}_i)$  such that  $\exp_i \circ \gamma = \exp_i$  on  $B'$ , where  $B' := B(O, \pi/2\Lambda) \subset \mathbb{R}^n$ . Now  $G_i$  forms a local pseudogroup defined as follows: For  $\gamma_1, \gamma_2, \gamma_3 \in G_i$ ,  $\gamma_1 \gamma_2 = \gamma_3$  holds if the composition  $\gamma_1 \circ \gamma_2$  is well-defined and coincides with  $\gamma_3$  in a neighborhood of the origin  $O$ .

We define the limit group germ  $G$  of  $G_i$  as follows: Let  $\mathcal{L}$  denote the set of all continuous maps  $f : B' \rightarrow B$  such that

$$\frac{1}{2} \leq \frac{d_0(f(x), f(y))}{d_0(x, y)} \leq 2, \quad (20)$$

equipped with the uniform topology, where  $d_0$  is the distance induced from  $\tilde{h}_0$ . By Ascoli-Arzelà's theorem,  $\mathcal{L}$  is compact. Therefore, passing to a subsequence, we may assume that  $G_i$  converges to a closed subset  $G$  with respect to the Hausdorff distance in  $\mathcal{L}$ . We see that  $G$  is a local pseudogroup consisting of isometric embeddings  $g : (B', \tilde{h}_0) \rightarrow (B, \tilde{h}_0)$  and that  $(B, h_i, G_i)$  converges to  $(B, \tilde{h}_0, G)$  in the equivariant Gromov-Hausdorff topology. This means that there exist  $\varepsilon_i$ -approximations

$$\phi_i : (B, \tilde{h}_i) \rightarrow (B, \tilde{h}_0), \quad \psi_i : (B, \tilde{h}_0) \rightarrow (B, \tilde{h}_i)$$

with  $\lim \varepsilon_i = 0$ , and maps

$$\rho_i : G_i \rightarrow G, \quad \lambda_i : G \rightarrow G_i$$

such that for every  $x, y \in B$  and  $\gamma_i \in G_i$ ,  $\gamma \in G$ , the following hold

$$d_0(\phi_i(\gamma_i(x)), \rho_i(\gamma_i)(\phi_i(x))) < \varepsilon_i, \quad d_i(\psi_i(\gamma(x)), \lambda_i(\gamma)(\psi_i(x))) < \varepsilon_i, \quad (21)$$

when they make sense. Roughly speaking this shows that the pseudogroup action of  $G_i$  on  $(B, \tilde{h}_i)$  is close to that of  $G$  on  $(B, \tilde{h}_0)$ . In particular, the quotient space  $(B', \tilde{h}_i)/G_i = B(p_i, 1/2)$  converges to  $(B', \tilde{h}_0)/G$ , which implies that

$$(B', \tilde{h}_0)/G = B(q, 1/2). \quad (22)$$

(See [28] for further details on basic properties of the equivariant Gromov-Hausdorff convergence.)



Let

$$\pi_i : B' \rightarrow B(q_i, 1/2), \quad \pi : B' \rightarrow B(q, 1/2),$$

be the natural projections.

**Example 2.1** Take  $M_i = \mathbb{S}_1 \times \mathbb{S}_{1/i}$ ,  $i = 1, 2, \dots$ , with the usual product metric. Then  $M_i$  collapses to  $\mathbb{S}_1$ , as  $i \rightarrow \infty$ , in the Gromov-Hausdorff distance. The map  $\gamma_k$  defined by  $\gamma_k(x, y) = (x, y + \frac{2\pi k}{i})$  on  $B(0, 1/2) \subset \mathbb{R}^2$  for any  $k \in \mathbb{Z}$  with  $|k| \leq \frac{i}{4\pi}$  belongs to  $G_i$ . Namely  $G_i$  identifies points  $(x, y)$  and  $(x', y')$  if  $x - x' = 0$ ,  $y - y' = \frac{2\pi k}{i}$ , where we use coordinates  $(x, y)$  for points in  $\mathbb{R}^2$ . Thus  $G_i$  acts like a covering transformation group.

Fukaya [25] proved that there exists a Lie group  $\widehat{G}$  containing  $G$  as an open subset.

When  $d = \dim X \leq n - 1$ , or equivalently when  $\dim G \geq 1$ , we say that  $M_i$  collapses to  $X$ , and that the collapsing is  $k$ -dimensional for  $k = n - d$ .

Let  $X$  be any element of the GH-closure  $\overline{\mathfrak{M}}_p(n, \Lambda, D)$  of  $\mathfrak{M}_p(n, \Lambda, D)$ . From (22),  $X$  can be locally described as orbit spaces. Therefore, it has the stratification

$$X = S_0(X) \supset S_1(X) \supset \dots \supset S_d(X), \quad (23)$$

such that, if  $S_j(X) \setminus S_{j+1}(X)$  is non-empty, it is a  $(d - j)$ -dimensional Riemannian manifold.

Actually  $S_j(X) \setminus S_{j+1}(X)$  is defined as the set of all point  $q \in X$  such that the tangent cone  $K_q(X)$  of  $X$  at  $q$  (see [11], [57]) is isometric to a product of the form  $\mathbb{R}^{d-j} \times Y^j$ , where  $Y$  has no nontrivial  $\mathbb{R}$ -factor.

Let us denote by

$$S(X) := S_1(X), \quad X^{reg} := X \setminus S(X),$$

the singular and the regular set of  $X$  respectively.

**Lemma 2.2** *Let  $X$  be any element of  $\overline{\mathfrak{M}}_p(n, \Lambda, D)$  which is not a point. Then*

- (1)  $X^{reg}$  is a  $C_*^2$ -Riemannian manifold;

- (2) For any compact subset  $K \subset X^{reg}$ , there exists a positive number  $i_K > 0$  such that  $\text{inj}(q) \geq i_K$  for all  $q \in K$ , where  $\text{inj}(q)$  denotes the injectivity radius at  $q$ ;
- (3)  $X^{reg}$  is convex in  $X$ . Namely, every geodesic joining two points in  $X^{reg}$  is contained in  $X^{reg}$ .

**Proof.** Let  $X$  be the limit of  $(M_i, h_i) \in \mathfrak{M}(n, \Lambda, D)$ . By [25], there exist Riemannian metric  $h_i^\varepsilon$  on  $M_i$  such that,  $h_i^\varepsilon \rightarrow h_i$  in the  $C^{1,\alpha}$ -topology as  $\varepsilon \rightarrow 0$ , for any  $0 < \alpha < 1$ , and

- (1)  $(M_i, h_i^\varepsilon)$  converges to  $X^\varepsilon$ , as  $i \rightarrow \infty$ , with respect to the Gromov-Hausdorff distance;
- (2) the regular part  $(X^\varepsilon)^{reg}$  of  $X^\varepsilon$  is a Riemannian manifold of class  $C^\infty$ ;
- (3)  $X^\varepsilon$  is  $\varepsilon$ -isometric to  $X$ . Namely, there exists a bi-Lipschitz map  $f^\varepsilon : X \rightarrow X^\varepsilon$  satisfying

$$\left| \frac{d_\varepsilon(f^\varepsilon(x), f^\varepsilon(y))}{d_X(x, y)} - 1 \right| < \varepsilon, \quad d_\varepsilon = d_{M_\varepsilon}.$$

More precisely, the norm of the  $k$ -th covariant derivatives of the curvature tensor  $R_{h_i^\varepsilon}$  of  $h_i^\varepsilon$  has the following uniform bound

$$\|\nabla^k R_{h_i^\varepsilon}\| \leq C(n, k, \varepsilon),$$

for any fixed  $k$  and  $\varepsilon$ . (See [6]).

Since  $X$  and  $X_\varepsilon$  have orbit-type singularities, the above (3) implies that  $f^\varepsilon(S_j(X)) = S_j(X^\varepsilon)$  for small  $\varepsilon$ , where  $S_j(X)$  are the stratification of  $X$  in (23). In particular,  $f^\varepsilon(X^{reg}) = (X^\varepsilon)^{reg}$ .

For a small  $\delta > 0$ , let  $V$  be the  $\delta$ -neighborhood of  $K$  in  $X$  such that  $\overline{V} \subset X^{reg}$ , and set  $V^\varepsilon := f^\varepsilon(V)$ . Then,

$$-1 \leq R(X^\varepsilon)|_{V^\varepsilon} \leq C(n). \tag{24}$$

(see Theorem 0.9 in [25]). Note that, by (3),  $\text{diam}(V^\varepsilon) \leq C_1$ ,  $\text{vol}(V^\varepsilon) \geq C_2$  for some uniform constants  $C_j = C_j(K) > 0$ ,  $j = 1, 2$ , which are independent

of  $\varepsilon$ . Thus, (24) together with Cheeger's theorem [14] implies that there is a positive number  $i_K < \delta$  independent of  $\varepsilon$  such that

$$\text{inj}_{X^\varepsilon}(q) \geq i_K, \quad (25)$$

for all  $q \in K^\varepsilon := f^\varepsilon(K)$ . Now a standart argument using the Cheeger-Gromov compactness applied to the convergence  $V^\varepsilon \rightarrow V$  implies that

$$\text{inj}_X(q) \geq i_K, \quad (26)$$

for all  $q \in K$ . It follows from (24), (25) and [2] that the metric of  $V$  and hence of  $X^{\text{reg}}$  is of class  $C_*^2$ .

Let us explain the basic idea of the estimate (26): Fix any point  $q \in K$  and two different direction  $\xi_1, \xi_2$  of  $X$  at  $q$ . First, we can prove that there exist geodesics  $\gamma_j^\varepsilon$ ,  $j = 1, 2$ , of length  $i_K$  starting at  $q^\varepsilon := f^\varepsilon(q)$  such that  $(f^\varepsilon)^{-1}(\gamma_j^\varepsilon)$  converge to geodesics  $\gamma_j$  of length  $i_K$  tangent to  $\xi_j$  as  $\varepsilon \rightarrow 0$ . This follows from the local extendability of geodesics in  $V$  as follows.

We use Alexandrov geometry. Denote by  $\Sigma_q(X)$  the space of directions of  $X$  at  $q$ , which is isometric to the  $(d-1)$ -dimensional unit sphere in the present case. Put  $\xi := \xi_j \in \Sigma_q(X)$ ,  $j = 1, 2$ , for simplicity, and take  $x_\varepsilon \in V$  such that the angle between  $\xi$  and the direction  $\xi_\varepsilon$  determined by a geodesic from  $q$  to  $x_\varepsilon$  tends to 0 as  $\varepsilon \rightarrow 0$ . Now consider the geodesic  $\gamma^\varepsilon$  starting from  $q^\varepsilon$  through  $f^\varepsilon(x_\varepsilon)$  of length  $i_K$ . Then it is easily verified that  $(f^\varepsilon)^{-1}(\gamma^\varepsilon)$  converges to a geodesic whose direction coincides with  $\xi$ .

Now let  $\tilde{\gamma}_j^\varepsilon(s)$ ,  $j = 1, 2$ , be geodesics on the complete simply connected surface  $M_{C(n)}^2$  of constant curvature  $C(n)$  starting at a point  $O$  with

$$\angle_{q^\varepsilon}(\gamma_1^\varepsilon, \gamma_2^\varepsilon) = \angle_0(\tilde{\gamma}_1^\varepsilon, \tilde{\gamma}_2^\varepsilon).$$

By Toponogov's comparison theorem,

$$d_\varepsilon(\gamma_1^\varepsilon(s), \gamma_2^\varepsilon(t)) \geq \tilde{d}(\tilde{\gamma}_1^\varepsilon(s), \tilde{\gamma}_2^\varepsilon(t)), \quad 0 \leq s, t \leq i_K, \quad (27)$$

where  $\tilde{d}$  is the standard distance on  $M_{C(n)}^2$ . It follows that

$$d_X(\gamma_1(s), \gamma_2(t)) \geq \tilde{d}(\tilde{\gamma}_1(s), \tilde{\gamma}_2(t)), \quad 0 \leq s, t \leq i_K,$$

where  $\tilde{\gamma}_j(s)$ ,  $j = 1, 2$ , are the geodesics on  $M_{C(n)}^2$  starting at  $O$  with

$$\angle_q(\xi_1, \xi_2) = \angle_0(\tilde{\gamma}_1, \tilde{\gamma}_2).$$

This yields  $\text{inj}_X(q) \geq i_K$  as required.

(3) of the lemma is a direct consequence of [53].

QED

The space  $X^\varepsilon$  which appears in the proof of Lemma 2.2 is called a *smooth element* in [25].

The geometric structure of the limit space  $X$  can be described in the following two ways:

**Theorem 2.3** *For every  $q \in X$ , let  $G$  be local pseudogroup defined as above, and set  $\ell := n - \dim(G \cdot O)$ , where  $G \cdot O$  denotes the orbit  $G(O)$ .*

- (1) *There exist a neighborhood  $U$  of  $q$ , a compact Lie group  $G_q$  and a faithful representation of  $G_q$  into the orthogonal group  $O(\ell)$ , a  $G_q$ -invariant smooth metric on a neighborhood  $V$  of  $O$  in  $\mathbb{R}^\ell$  such that  $U$  is bi-Lipschitz homeomorphic to  $V/G_q$ ;*
- (2) *There exists a  $C_*^2$ -Riemannian manifold  $Y$  with  $\dim Y = \dim X + \dim O(n)$  on which  $O(n)$  acts as isometries in such a way that*
  - (a)  *$X$  is isometric to  $Y/O(n)$ . Let  $\pi : Y \rightarrow X$  be the projection ;*
  - (b) *For every  $q \in X$  and  $\bar{q} \in \pi^{-1}(q)$ , the isotropy group*

$$H_{\bar{q}} := \{g \in O(n) \mid g(\bar{q}) = \bar{q}\}$$

*is isomorphic to  $G_q$ , where  $G_q$  is as in (1).*

**Proof.** (1) Take an  $\ell$ -dimensional disk  $V$  in  $B'$  which transversally meets the orbit  $G \cdot O$  at  $O$  and is invariant under the action of the isotropy group  $G_q := \{g \in G \mid g(O) = O\}$ . Fix a  $G_q$ -invariant metric on  $V$ . Then  $V/G_q$  is bi-Lipschitz homeomorphic to a neighborhood of  $q$ .

(2) We only describe the construction of the space  $Y$  below. Let  $FM_i$  denote the orthonormal frame bundle of  $M_i$  endowed with the natural Riemannian metric, which has uniformly bounded sectional curvature and diameter. Note that  $O(n)$  isometrically acts on  $FM_i$ . Passing to a subsequence, we may assume that  $(FM_i, O(n))$  converges to  $(Y, O(n))$  in the equivariant GH-topology, where, as shown below,  $Y$  is a Riemannian manifold on which  $O(n)$  act isometrically. It follows that  $X = Y/O(n)$ .

To show that  $Y$  is a Riemannian manifold. Let  $B' \subset B \subset \mathbb{R}^n$ ,  $(B, \tilde{h}_i, G_i)$  and  $(B, \tilde{h}_0, G)$  be as described earlier, so that  $(B', \tilde{h}_i)/G_i = B(q_i, \pi/2\Lambda)$  and  $(B', \tilde{h}_0)/G = B(q_0, \pi/2\Lambda)$ . The pseudogroup action of  $G_i$  on  $(B', \tilde{h}_i)$  induces an isometric pseudogroup action, denoted by  $\hat{G}_i$ , on the frame bundle  $F(B', \tilde{h}_i)$  of  $(B', \tilde{h}_i)$  defined by differential. Therefore,  $F(B', \tilde{h}_i)/\hat{G}_i = FB(q_i, \pi/2\Lambda)$ . Passing to a subsequence, we may assume that  $(F(B', \tilde{h}_i), \hat{G}_i)$  converges to  $(F(B', \tilde{h}_0), \hat{G})$  in the equivariant Gromov-Hausdorff topology, where  $\hat{G}$  denotes the isometric pseudogroup action on  $F(B', \tilde{h}_0)$  induced from that of  $G$  on  $(B', \tilde{h}_0)$ . The action of  $\hat{G}$  on  $F(B', \tilde{h}_0)$  is free. Indeed, let  $\hat{g} \in \hat{G}$  satisfies

$$g(x) = x, \quad dg_x(o) = o, \quad o \in O_x(n), \quad x \in B'.$$

Then  $g$  is an isometry with  $g(x) = x$ ,  $dg_x = \text{id}$  on  $T_x B'$ , so that  $g = \text{id}$  on  $B'$ , see [51]. Therefore,  $F(B', \tilde{h}_0)/\hat{G}$  is a Riemannian manifold, and so is  $Y$ .

Next let us show that  $\dim Y = \dim O(n) + d$ ,  $d = \dim X$  or, equivalently, that the action of  $O(n)$  on  $Y$  is fathfull. Let  $x \in X^{reg} \cap B(q, \pi/2\Lambda)$ . It is easy to see that the pseudogroup action of  $G$  on  $(B, \tilde{h}_0) \cap \pi^{-1}(X^{reg} \cap B(q, \pi/2\Lambda))$  is free, and that  $\pi^{-1}(x)$  is isometric to  $O(n)$ , where  $\pi : B' \rightarrow B(q, \pi/2\Lambda)$  is the natural projection. This shows that  $\dim Y = \dim O(n) + d$ . QED.

In the future, we make an extensive use of Theorem 2.3, especially representation (2).

Note that  $q$  is an orbifold point if and only if  $G_q \simeq H_{\bar{q}}$  is finite. This actually occurs for every  $q \in X$  in the case of collapsing being one dimensional:

**Corollary 2.4** ([26]) *If  $\dim X = n - 1$ , then  $X$  is an orbifold.*

**Proof.** By Theorem 2.3, for every  $q \in X$ , we can take a neighborhood  $U$  of  $q$ , a compact Lie group  $G_q \subset O(\ell)$ , a  $G_q$ -invariant metric of class  $C_*^2$  on an neighborhood  $V$  of  $O$  in  $\mathbb{R}^\ell$ , such that  $U$  is bi-Lipschitz homeomorphic to  $V/G_q$ . Since  $n - 1 \leq \ell < n$ , we have  $\ell = n - 1$ . It follows that  $\dim G_q = \dim V - \dim U = 0$ . Thus  $G_q$  is finite, and therefore  $q$  is an orbifold point.

## 2.2 Fiber bundle theorems

**Theorem 2.5** ([25], Theorem 10.1) *Suppose a sequence  $X_i$  in  $\overline{\mathfrak{M}}_p(n, \Lambda, D)$  converges to  $X$  with respect to the Gromov-Hausdorff distance. Then there*

are  $O(n)$ -Riemannian manifolds  $Y_i$  and  $Y$  of class  $C_*^2$  and  $O(n)$ -maps  $\tilde{f}_i : Y_i \rightarrow Y$  and maps  $f_i : X_i \rightarrow X$  such that

- (1)  $X_i = Y_i/O(n)$ ,  $X = Y/O(n)$ . Let  $\pi_i : Y_i \rightarrow X_i$ ,  $\pi : Y \rightarrow X$  be the projections;
- (2)  $\tilde{f}_i$  are  $\varepsilon_i$ -Riemannian submersions as well as  $\varepsilon_i$ -approximations, where  $\lim \varepsilon_i = 0$ . Namely,  $\tilde{f}_i$  satisfies

$$e^{-\varepsilon_i} < \frac{|d\tilde{f}_i(\xi)|}{|\xi|} < e^{\varepsilon_i}, \quad (28)$$

for all tangent vectors  $\xi$  orthogonal to fibers of  $\tilde{f}_i$ ;

- (3)  $f \circ \pi_i = \pi \circ \tilde{f}_i$ ;
- (4) for every  $y \in Y$ , the isotropy subgroup  $\{g \in O(n) \mid g(y) = y\}$  is isomorphic to  $G_{\pi(y)}$ , where  $G_{\pi(y)}$  is as in Theorem 2.3.

We shall call the maps  $f_i : X_i \rightarrow X$  regular  $\varepsilon_i$ -approximations for simplicity.

**Corollary 2.6** Suppose a sequence  $X_i$  in  $\overline{\mathfrak{M}}_p(n, \Lambda, D)$  converges to  $X$  with respect to the Gromov-Hausdorff distance.

- (1) If both  $X_i$  and  $X$  have no singularities, then the map  $f_i : X_i \rightarrow X$  is an  $\varepsilon_i$ -Riemannian submersion;
- (2) If  $\dim X_i = \dim X$ , then the map  $f_i : X_i \rightarrow X$  is an  $\varepsilon_i$ -almost isometry. Namely, it satisfies, for every  $x, y \in X_i$ ,

$$e^{-\varepsilon_i} < \frac{d_i(f_i(x), f_i(y))}{d_X(x, y)} < e^{\varepsilon_i}, \quad d_i = d_{X_i}.$$

**Corollary 2.7** Suppose a sequence  $X_i$  in  $\overline{\mathfrak{M}}(n, \Lambda, D)$  converges to  $X$  with respect to the Gromov-Hausdorff distance. If  $\dim X = \dim X_i - 1$  for all  $i$ , then the map  $f_i : X_i \rightarrow X$  is a Seifert  $S^1$ -bundle. Namely for any  $q \in X$  there exist a neighborhood  $V$  of  $q$ , an open set  $U \subset \mathbb{R}^{n-1}$  and a finite group  $G$  acting on both  $U$  and  $S^1$  such that

- (1) there is a bi-Lipschitz homeomorphism  $\mathcal{H} : U/G \rightarrow V$ ;

- (2) Consider the diagonal  $G$ -action on  $U \times S^1$ . Then we have the following commutative diagram:

$$\begin{array}{ccc}
U \times S^1 & \xrightarrow{\pi} & U \\
\pi \downarrow & & \downarrow \pi_i \\
U \times S^1 / G & \xrightarrow{\pi_*} & U / G \\
\mathcal{H}_i \downarrow & & \downarrow \mathcal{H} \\
f_i^{-1}(V) & \xrightarrow{f_i} & V,
\end{array}$$

where  $\pi$  and  $\pi_i$  denote the projections and  $\mathcal{H}_i$  is a fiber-preserving isomorphism.

## 2.3 Measured Gromov-Hausdorff distance

Let  $(X_i, \mu_i)$  and  $(X, \mu)$  be probability Borel measures on compact metric spaces  $X_i$  and  $X$ . Fukaya defined the notion of measured Gromov-Hausdorff convergence  $(X_i, \mu_i) \rightarrow (X, \mu)$ , or weak convergence in short. By definition, this is the case when there are measurable  $\varepsilon_i$ -approximations  $\psi_i : X_i \rightarrow X$  with  $\lim \varepsilon_i = 0$ , see (19), such that the pushforward measure  $(\psi_i)_* \mu_i$  weakly converges to  $\mu$  in the usual sense. Namely,  $\int_{X_i} f \circ \psi_i d\mu_i \rightarrow \int_X f d\mu$  as  $i \rightarrow \infty$  for any  $f \in C(X)$ , where  $C(X)$  denotes the space of continuous functions on  $X$ . In this subsection, we define the measured Gromov-Hausdorff “distance” that provides a topology equivalent to the weak topology.

**Definition 2.8** Let  $(X, \mu)$  and  $(X', \mu')$  be probability Borel measures on compact metric spaces  $X$  and  $X'$ .

The measured Gromov-Hausdorff “distance”  $d_{mGH}((X, \mu), (X', \mu'))$  between  $X$  and  $X'$  is defined as the infimum of those  $\varepsilon > 0$  that there are measurable  $\varepsilon$ -approximations  $\psi : X \rightarrow X'$  and  $\psi' : X' \rightarrow X$  satisfying

$$\mu(\psi^{-1}(A')) < \mu'((A')^\varepsilon) + \varepsilon, \quad \mu'((\psi')^{-1}(A)) < \mu(A^\varepsilon) + \varepsilon, \quad (29)$$

for all Borel sets  $A \subset X$  and  $A' \subset X'$ , where  $A^\varepsilon$  denotes the  $\varepsilon$ -neighborhood of  $A$ :  $A^\varepsilon := \{x \in X : d_X(x, A) < \varepsilon\}$ .

**Remark 2.9**  $d_{mGH}$  is a generalization of the Prohorov metric on the space of probability measures on a fixed space (see [9]).

**Lemma 2.10** We have almost triangle inequality:

$$\begin{aligned} d_{mGH}((X_1, \mu_1), (X_2, \mu_2)) \\ \leq 2(d_{mGH}((X_1, \mu_1), (X_3, \mu_3)) + d_{mGH}((X_3, \mu_3), (X_2, \mu_2))). \end{aligned}$$

**Proof.** Let  $d_{ij} := d_{mGH}((X_i, \mu_i), (X_j, \mu_j))$ , and  $d := d_{13} + d_{32}$ . By definition, for any  $\varepsilon > 0$  and for any  $i, j \in \{1, 3\}$  or  $i, j \in \{3, 2\}$ , there are  $(d_{ij} + \varepsilon/2)$ -approximation  $\psi_{ji} : X_i \rightarrow X_j$ , satisfying

$$\mu_i(\psi_{ji}^{-1}(A_j)) < \mu_j(A_j^{d_{ij} + \varepsilon/2}) + d_{ij} + \varepsilon/2,$$

for any Borel set  $A_j \subset X_j$ . Define  $\psi_{21} : X_1 \rightarrow X_2$  by  $\psi_{21} := \psi_{23} \circ \psi_{31}$ , which is a  $2(d + \varepsilon)$ -approximation. Then, for any Borel set  $A_2 \subset X_2$ ,

$$\begin{aligned} \mu_1(\psi_{21}^{-1}(A_2)) &= \mu_1(\psi_{31}^{-1}\psi_{23}^{-1}(A_2)) < \mu_3((\psi_{23}^{-1}(A_2))^{d_{13} + \varepsilon/2}) + d_{13} + \varepsilon/2 \\ &< \mu_3(\psi_{23}^{-1}((A_2)^{d_{13} + d_{32} + \varepsilon})) + d_{13} + \varepsilon/2 \\ &< \mu_2((A_2)^{d + d_{32} + \varepsilon}) + d + \varepsilon. \end{aligned}$$

Similarly, for  $\psi_{12} := \psi_{13} \circ \psi_{32}$  we have  $\mu_2(\psi_{12}^{-1}(A_1)) < \mu_1((A_1)^{d + d_{31} + \varepsilon}) + d + \varepsilon$ , for any Borel set  $A_1 \subset X_2$ , and, therefore, the lemma follows. QED.

**Lemma 2.11**  $d_{mGH}((X, \mu), (X', \mu')) = 0$  if and only if there exists an isometry  $\psi : X \rightarrow X'$  such that  $\psi_*(\mu) = \mu'$ .

**Proof.** Suppose  $d_{mGH}((X, \mu), (X', \mu')) = 0$ . By definition, there are  $\varepsilon_i$ -approximations  $\psi_i : X \rightarrow X'$  with  $\lim \varepsilon_i = 0$  such that

$$\mu(\psi_i^{-1}(A')) < \mu'((A')^{\varepsilon_i}) + \varepsilon_i,$$

for every closed subset  $A' \subset X'$ . As  $X, X'$  are compact, we may assume, using (19), that  $\psi_i$  uniformly converges to an isometry  $\psi : X \rightarrow X'$ . Since  $\psi^{-1}(A') \subset (\psi_i^{-1}(A'))^{\delta_i}$  for some  $\delta_i \rightarrow 0$ , it follows that

$$\mu(\psi^{-1}(A')) \leq \mu((\psi_i^{-1}(A'))^{\delta_i}) \leq \mu(\psi_i^{-1}((A')^{\delta_i + \varepsilon_i})) \leq \mu'((A')^{\delta_i + 2\varepsilon_i}) + \varepsilon_i.$$



Letting  $i \rightarrow \infty$ , we obtain  $\mu(\psi^{-1}(A')) \leq \mu'(A')$ . Taking complement, we have  $\mu(\psi^{-1}(U')) \geq \mu'(U')$  for any open set  $U' \subset X'$ . It follows that  $\mu(\psi^{-1}((A')^\varepsilon)) \geq \mu'((A')^\varepsilon)$ . Letting  $\varepsilon \rightarrow 0$ , we obtain  $\mu(\psi^{-1}(A')) \geq \mu'(A')$ . Thus, we have  $\mu(\psi^{-1}(A')) = \mu'(A')$  for every closed set  $A'$  and hence for every Borel subset  $A'$ . This completes the proof of the lemma. QED.

**Proposition 2.12** *A sequence  $(X_i, \mu_i)$  weakly converges to  $(X, \mu)$  if and only if  $\lim d_{mGH}((X_i, \mu_i), (X, \mu)) = 0$ .*

**Proof.** Take  $\varepsilon_i$ -approximations  $\psi_i : X_i \rightarrow X$  with  $\lim \varepsilon_i = 0$  such that  $\int_{X_i} f \circ \psi_i d\mu_i \rightarrow \int_X f d\mu$  for every  $f \in C(X)$ . First, using the weak convergence  $(\psi_i)_*\mu_i \rightarrow \mu$ , we show by contradiction that, for any Borel set  $A \subset X$ ,

$$((\psi_i)_*\mu_i)(A) < \mu(A^{\varepsilon'_i}) + \varepsilon'_i, \quad (30)$$

$$\mu(A) < ((\psi_i)_*\mu_i)(A^{\varepsilon'_i}) + \varepsilon'_i, \quad (31)$$

for some  $\varepsilon'_i \rightarrow 0$ . Suppose (30) does not hold. Then there are closed Borel sets  $A_i$  of  $X$  such that

$$((\psi_i)_*\mu_i)(A_i) \geq \mu(A_i^c) + c, \quad (32)$$

for some constant  $c > 0$  independent of  $i$ . We may assume that  $A_i$  converges to a closed set  $A$  with respect to the Hausdorff distance in  $X$ . Take  $\varepsilon_2 > \varepsilon_1 > 0$  with  $A_i \subset A^{\varepsilon_1} \subset A^{\varepsilon_2} \subset A_i^c$  for sufficiently large  $i$ . Choose  $f \in C(X)$  such that  $0 \leq f \leq 1$ ,  $f = 1$  on  $A^{\varepsilon_1}$ , and  $\text{supp}(f) \subset A^{\varepsilon_2}$ . Then

$$\begin{aligned} \mu(A_i^c) \geq \mu(A^{\varepsilon_2}) &\geq \int_X f d\mu = \lim_{i \rightarrow \infty} \int_{X_i} f \circ \psi_i d\mu_i \\ &\geq \limsup_{i \rightarrow \infty} ((\psi_i)_*\mu_i)(A^{\varepsilon_1}) \geq \limsup_{i \rightarrow \infty} ((\psi_i)_*\mu_i)(A_i). \end{aligned}$$

This is a contradiction to (32).

Next suppose (31) does not hold. Then for some Borel sets  $A_i$  of  $X$  we have

$$\mu(A_i) \geq ((\psi_i)_*\mu_i)(A_i^c) + c, \quad (33)$$

for some constant  $c > 0$  independent of  $i$ . Let  $A_i \subset A^{\varepsilon_1} \subset A^{\varepsilon_2} \subset A_i^c$  and  $f \in C(X)$  be given as above. Then,

$$\begin{aligned} \liminf_{i \rightarrow \infty} ((\psi_i)_*\mu_i)(A_i^c) &\geq \liminf_{i \rightarrow \infty} ((\psi_i)_*\mu_i)(A^{\varepsilon_2}) \\ &\geq \lim_{i \rightarrow \infty} \int_{X_i} f \circ \psi_i d\mu_i = \int_X f d\mu \geq \mu(A^{\varepsilon_1}) \geq \mu(A_i). \end{aligned}$$

This is a contradiction to (33).

Let  $\psi'_i : X \rightarrow X_i$  be any measurable  $\varepsilon_i$ -approximation such that  $d_i(\psi'_i \circ \psi_i(x_i), x_i) < \varepsilon_i$  for every  $x_i \in X_i$ ,  $d(\psi_i \circ \psi'_i(x), x) < \varepsilon_i$  for every  $x \in X$ . Using (31), we obtain, for any Borel  $A_i \subset X_i$ ,

$$\begin{aligned} \mu((\psi'_i)^{-1}(A_i)) &< \mu_i\left(\psi_i^{-1}\left[\left((\psi'_i)^{-1}(A_i)\right)^{\varepsilon'_i}\right]\right) + \varepsilon'_i \\ &< \mu_i(A_i^{2\varepsilon_i + \varepsilon'_i}) + \varepsilon'_i. \end{aligned}$$

Together with (30), we have  $d_{mGH}((X_i, \mu_i), (X, \mu)) < 2\varepsilon_i + \varepsilon'_i$ .

Finally we shall prove the converse. Since  $\mu_i, \mu$  are probability measures, shifting  $f \in C(X)$ ,  $f \mapsto f - \min(f)$ , and normalising it,  $f \mapsto f / \max(f)$ , we may assume  $0 \leq f \leq 1$ . Take a large positive integer  $k$ , and set  $A_j := \{x \in X \mid f(x) \geq j/k\}$  for  $0 \leq j \leq k$ . It is straightfoward to see that, for any Borel measure  $\mu$  on  $X$ ,

$$\frac{1}{k} \sum_{j=1}^k \mu(A_j) \leq \int_X f d\mu \leq \frac{1}{k} + \frac{1}{k} \sum_{j=1}^k \mu(A_j). \quad (34)$$

Now take  $\varepsilon_i$ -approximation  $\psi_i : X_i \rightarrow X$  with  $\varepsilon_i \rightarrow 0$  such that  $((\psi_i)_* \mu_i)(A) < \mu(A^{\varepsilon_i}) + \varepsilon_i$  for every closed set  $A \subset X$ , where  $\varepsilon_i \rightarrow 0$ . Letting  $i \rightarrow \infty$ , we have  $\limsup((\psi_i)_* \mu_i)(A) \leq \mu(A)$ . Therefore, in the above situation, we obtain  $\limsup_{i \rightarrow \infty}((\psi_i)_* \mu_i)(A_j) \leq \mu(A_j)$  for each  $0 \leq j \leq k$ . It follows from (34) that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \int_X f d((\psi_i)_* \mu_i) &\leq \frac{1}{k} + \sum_{j=1}^k \limsup_{i \rightarrow \infty} ((\psi_i)_* \mu_i)(A_j) \\ &\leq \frac{1}{k} + \sum_{j=1}^k \mu(A_j) \leq \frac{1}{k} + \int_X f d\mu. \end{aligned}$$

Thus, letting  $k \rightarrow \infty$ ,

$$\limsup_{i \rightarrow \infty} \int_X f d((\psi_i)_* \mu_i) \leq \int_X f d\mu.$$

Replacing  $f$  by  $1 - f$ , we get  $\liminf_{i \rightarrow \infty} \int_X f d((\psi_i)_* \mu_i) \geq \int_X f d\mu$ , and

$$\lim_{i \rightarrow \infty} \int_X f d((\psi_i)_* \mu_i) = \int_X f d\mu,$$

as required.

QED

Finally, for pointed metric measure spaces  $(X, p, \mu)$  and  $(X', p', \mu')$ , we define the pointed measured Gromov-Hausdorff distance,

$$d_{pmGH}((X, p, \mu), (X', p', \mu')),$$

as the infimum of those  $\varepsilon > 0$  that there are  $\varepsilon$ -approximations  $\psi : B(p, 1/\varepsilon) \rightarrow B(p', 1/\varepsilon)$  and  $\psi' : B(p', 1/\varepsilon) \rightarrow B(p, 1/\varepsilon)$  with  $\psi(p) = p'$ ,  $\psi'(p') = p$ , satisfying (29), for all Borel sets  $A \subset B(p, 1/\varepsilon)$  and  $A' \subset B(p', 1/\varepsilon)$ .

### 3 Smoothness of the density functions

Recall that for a compact Riemannian manifold  $(M, h)$ ,  $\mu_M$  denotes the normalized measure,

$$\mu_M := dV_h / \text{Vol}(M),$$

where  $dV$  is the Riemannian measure of  $M$ . In this section, we consider the set  $\mathfrak{MM}_p(n, \Lambda, D)$  consisting of all pointed Riemannian manifolds with measure  $(M, p, \mu_M)$ , where  $M \in \mathfrak{M}_p(n, \Lambda, D)$ . Fukaya [24] proved the pre-compactness of  $\mathfrak{MM}_p(n, \Lambda, D)$  with respect to the pointed measured Gromov-Hausdorff topology.

Let us consider a sequence  $(M_i, p_i)$  in  $\mathfrak{M}_p(n, \Lambda, D)$  converging to  $(X, p) \in \overline{\mathfrak{M}}_p(n, \Lambda, D)$ . Let  $\varphi_i : M_i \rightarrow X$  be a measurable  $\varepsilon_i$ -approximation with  $\lim \varepsilon_i = 0$ . Passing to a subsequence, we may assume that  $(M_i, p_i, \mu_i)$  converges to some  $(X, p, \mu)$  in the pointed measured Gromov-Hausdorff topology. Here  $\mu$  is some probability measure of  $X$ . (See [24]). More precisely, the pushforward measure  $(\varphi_i)_*(\mu_i)$  weak\* sub-converges to  $\mu$ .

The following lemma is known in [24]:

**Lemma 3.1** (1)  $\mu(S(X)) = 0$ , where  $S(X)$  denotes the singular set of  $X$ ;

(2) there exists a density function  $\rho_X$  on  $X$  such that

(a)  $\mu = \rho_X \mu_X$ , where  $\mu_X$  denotes the normalized Riemannian volume element of  $X$ ;

(b)  $\tilde{S}(X) = \{q \in X \mid \rho_X(q) = 0\}$ , where  $\tilde{S}(X)$  is the set of all points  $q$  of  $S(X)$  which are not orbifold points.

In this section, we discuss some properties of  $\rho_X$  concerning the smoothness, and prove the following:

**Lemma 3.2**  $\rho_X$  is of class  $C_*^2$  on  $X^{reg}$ .

Concerning Lemma 3.2, Kasue proved in [38] that  $\rho_X$  is of class  $C^{1,\alpha}$  for any  $0 < \alpha < 1$ . The method in [38] is to use smooth approximations of the metric of  $M_i$  as in the proof of Lemma 2.2. Our method discussed below is more direct and contains an extension of Montgomery and Zippin's result on the smoothness of isometric group actions.

First we consider the case when  $S(X)$  is empty, namely the case when  $X$  is a Riemannian manifold. In this case, we can approximate  $\varphi_i$  by an almost Riemannian submersion  $f_i : M_i \rightarrow X$  such that (see [24])

- (1) the pushforward measure  $(f_i)_*(\mu_i)$  weak\* converges to  $\mu$ ,
- (2)  $\text{Vol}(f_i^{-1}(q))/\text{Vol}(M_i)$  converges to  $\rho_X(q)/\text{Vol}(X)$  in the  $C^0$ -topology.

Fix  $q_0 \in X$  and put  $q_i := \psi_i(q_0)$ , where  $\psi_i : X \rightarrow M_i$  is an  $\epsilon_i$ -approximation such that  $d_X(f_i \circ \psi_i(x), x) < \epsilon_i$ .

Let  $B' \subset B \subset \mathbb{R}^n$ ,  $(B, \tilde{h}_i, G_i)$  and  $(B, \tilde{h}_0, G)$  be as in Section 2 so that  $(B', \tilde{h}_i)/G_i = B(q_i, 1/2)$  and  $(B', \tilde{h}_0)/G = B(q_0, 1/2)$ . Let

$$\pi_i : B' \rightarrow B(q_i, 1/2) \subset M_i, \quad \pi : B' \rightarrow B(q_0, 1/2) \subset X,$$

be the natural projections. Since  $X$  is a Riemannian manifold, the pseudogroup action of  $G$  on  $B'$  is free.

We now need to establish the following result on the smoothness of isometric group actions.

**Theorem 3.3** *Let  $G$  be a Lie group, and  $M$  a Riemannian manifold of class  $C_*^k$  with  $k \geq 1$ . Suppose that the action of  $G$  on  $M$  is isometric. Then the  $G$ -action on  $M$ ,*

$$G \times M \rightarrow M, \quad (g, x) \rightarrow gx,$$

*is of class  $C_*^{k+1}$ , where we consider the analytic structure on  $G$ .*

It is proved in Calabi-Hartman [12] and Shefel [56] that the transformation  $g : M \rightarrow M$  defined by each  $g \in G$  is of class  $C^{k,\alpha}$ . Theorem 3.3 is a generalization of Montgomery-Zippin [47], p. 212, where it is stated that the  $G$ -action on  $M$  is of class  $C^k$ . The proof of Theorem 3.3 is deferred to Appendix B.

Note that in our situation, the pseudo-group  $G$  can be extended to a nilpotent Lie group ([25]). It follows from Theorem 3.3 that the pseudo-group action of  $G$  on  $B$  is of class  $C_*^3$ .

Let  $d = \dim X$ ,  $m := \dim G$  with  $n = d + m$ , and take a  $d$ -dimensional  $C^\infty$ -submanifold  $Q$  of  $B$  which transversally meets the orbit  $G \cdot O$  at the origin  $O$ . Let  $s : U_0 \rightarrow Q$  be a smooth coordinate chart of  $Q$  around 0, where  $U_0$  is an open subset of  $\mathbb{R}^d$ . From Theorem 3.3 together with the inverse function theorem, taking  $Q$  smaller if necessary, we may assume that, for some neighborhood  $U$  of  $O$  in  $B$  and for a neighborhood  $G^*$  of the identity in  $G$ , the mapping

$$G^* \times U_0 \rightarrow U, (g, x) \rightarrow g(s(x)), \quad g \in G^*, x \in U_0, \quad (35)$$

gives  $C_*^3$ -coordinates in  $U$ .

We can consider every element  $X$  of the Lie algebra  $\mathfrak{g}$  of  $G$  as a Killing field on  $B'$  by setting

$$X(\mathbf{x}) := \frac{d}{dt}(\exp tX \cdot \mathbf{x})|_{t=0},$$

where  $\mathbf{x}$  denote points in  $U$ . Then, for any  $\mathbf{x} \in U$ , there is a unique  $g \in G^*$  and  $x \in U_0$  with  $\mathbf{x} = g(s(x))$ . We define

$$\tilde{\rho}(\mathbf{x}) = |Ad_g(X_1)(\mathbf{x}) \wedge \cdots \wedge Ad_g(X_m)(\mathbf{x})|,$$

where  $X_1, \dots, X_m$  be a basis of  $\mathfrak{g}$ , and the norm is taken with respect to  $\tilde{h}_0$ . Here the adjoint representation  $Ad : G \rightarrow GL(\mathfrak{g})$  is defined by

$$Ad_g(X) := \frac{d}{dt}(g \cdot \exp tX \cdot g^{-1})|_{t=0}.$$

We show that  $\tilde{\rho}$  is  $G$ -invariant and of class  $C_*^2$ . Let  $\mathbf{x} = g(s(x))$  and  $g' \in G^*$ . We denote by  $L_{g'}$  the left translation by  $g'$ . Since  $Ad_{g'g}(X_i)(g'\mathbf{x}) = (L_{g'})_* Ad_g(X_i)(\mathbf{x})$ , we have  $\tilde{\rho}(g'\mathbf{x}) = \tilde{\rho}(\mathbf{x})$  and  $\tilde{\rho}$  is  $G$ -invariant. Recall that the correspondence  $\mathbf{x} = g(s(x)) \rightarrow (g, x)$  is of class  $C_*^3$  from the inverse

function theorem. It follows that the map  $(t, \mathbf{x}) \rightarrow g(\exp tX_i)g^{-1}(\mathbf{x})$  is of class  $C_*^3$ . Thus  $\mathbf{x} \rightarrow Ad_g(X_i)(\mathbf{x})$  is of class  $C_*^2$ , and so is  $\tilde{\rho}(\mathbf{x})$ .

Since  $\tilde{\rho}$  is  $G$ -invariant, there is a function  $\rho$  defined on a neighborhood  $V = \pi(U)$  of the point  $q_0$  such that  $\tilde{\rho} = \rho \circ \pi$ . Obviously  $\rho$  is of class  $C_*^2$ .

We shall prove that  $\rho_X$  is of class  $C_*^2$  by showing that  $\rho_X(x)/\rho(x)$  is constant for  $x \in V$ . Basically we follow the argument in [27]. Let

$$\begin{aligned} G'_i &:= \{g \in G_i \mid d_{(B, \tilde{h}_i)}(g(O), O) < 1/2\}, \\ G' &:= \{g \in G \mid d_{(B, \tilde{h}_0)}(g(O), O) < 1/2\}. \end{aligned}$$

Let us consider a left-invariant Riemannian metric on  $G$ . First we need to show that

$$\frac{\text{Vol}(G'(s(x)))}{\rho(x)} = \text{const},$$

on a neighborhood of  $q_0$ . Define  $F^x : G' \rightarrow G'(s(x))$  by  $F^x(g) = g(s(x))$ . Let  $X_1, \dots, X_m$  be an orthonormal basis of  $\mathfrak{g}$ . For any  $g \in G'$ , we have

$$F_*^x(X_i(g)) = \frac{d}{dt}(g \exp tX_i(s(x)))_{t=0} = Ad_g(X_i)(F^x(g)).$$

Therefore,

$$\begin{aligned} \text{Vol}(G'(s(x))) &= \int_{G'} |Ad_g(X_1) \wedge \dots \wedge Ad_g(X_m)|(g(s(x))) \\ &= \int_{G'} \tilde{\rho}(g(s(x))) = \rho(x) \text{Vol}(G'). \end{aligned}$$

For the rest of the argument, we can go through along the same line as in [27], which we outline below.

Set

$$\begin{aligned} E_i(x, \delta) &:= \{\mathbf{y} \in B \mid \text{there exists } g \in G'_i \text{ such that } d_{\tilde{h}_i}(\mathbf{y}, g(s(x))) < \delta\}, \\ E_0(x, \delta) &:= \{\mathbf{y} \in B \mid \text{there exists } g \in G' \text{ such that } d_{\tilde{h}_0}(\mathbf{y}, g(s(x))) < \delta\}. \end{aligned}$$

Then one can check

$$\lim_{i \rightarrow \infty} \sup_{x \in V} \frac{\text{Vol}(E_i(x, \delta))}{\text{Vol}(E_0(x, \delta))} = 1, \quad (36)$$

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}(E_0(x, \delta))}{\delta^d} = \omega_d \text{Vol}(G'(s(x))), \quad (37)$$

where  $\omega_d$  denotes the volume of unit ball in  $\mathbb{R}^d$ . This implies

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}(E_0(x, \delta))}{\text{Vol}(E_0(x', \delta))} \frac{\rho(x')}{\rho(x)} = 1,$$

for all  $x, x' \in U$ . One can prove that there exists  $c > 0$  independent of  $x$  such that

$$\lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} \frac{\text{Vol}(E_i(x, \delta))}{\text{Vol}(G'_i) \delta^n \text{Vol}(f_i^{-1}(x))} = c. \quad (38)$$

These yield

$$\frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(f_i^{-1}(x'))} \frac{\rho(x')}{\rho(x)} = 1.$$

Since

$$\lim_{i \rightarrow \infty} \frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(M_i)} = \frac{\rho_X(x)}{\text{Vol}(X)},$$

we conclude

$$\frac{\rho(x')}{\rho(x)} \frac{\rho_X(x)}{\rho_X(x')} = 1,$$

which conclude that  $\rho_X$  is of class  $C_*^2$ .

Next consider the general case when  $X$  is not a Riemannian manifold. Since the above argument is local, it follows that  $\rho_X$  is of class  $C_*^2$  on  $X^{reg}$ . This completes the proof of Lemma 3.2. QED

Here we give the proof of Lemma 3.1, because the idea in the proof is used in later sections. We recall the characterization of  $\rho_X$  using orthonormal frame bundles.

Let  $FM_i$  denote the orthonormal frame bundle of  $M_i$  endowed with the natural Riemannian metric, which has uniformly bounded sectional curvature and diameter. Put

$$d\tilde{\mu}_i = \frac{dV_{FM_i}}{\text{Vol}(FM_i)}, \quad (39)$$

where  $dV_{FM_i}$  is the natural volume element on  $FM_i$  with  $O(n)$  acting isometrically on  $FM_i$ . Passing to a subsequence, we may assume that  $(FM_i, \tilde{\mu}_i, O(n))$  converges to  $(Y, \tilde{\mu}, O(n))$  in the equivariant measured Gromov-Hausdorff topology, where  $Y$  is a Riemannian manifold and  $\tilde{\mu}$  is a probability measure

on  $Y$  invariant under the  $O(n)$ -action. The notion of equivariant measured Gromov-Hausdorff topology is defined in a way similar to that of measured Gromov-Hausdorff topology, where a measurable  $\varepsilon_i$ -approximation maps  $\tilde{\psi}_i : FM_i \rightarrow Y$  with the property  $(\tilde{\psi}_i)_*(\tilde{\mu}_i) \rightarrow \tilde{\mu}$  is required to satisfy conditions similar to (21). By Theorem 2.5, there are  $\varepsilon_i$ -regular maps  $\tilde{f}_i : FM_i \rightarrow Y$  and  $f_i : M_i \rightarrow X$  such that  $f_i \circ \pi_i = \pi \circ \tilde{f}_i$ . We may replace  $\psi_i$  by  $\tilde{f}_i$ .

Note that the probability measure  $\tilde{\mu}$  on  $Y$  can be written as

$$d\tilde{\mu} = \rho_Y \frac{dV_{\tilde{h}}}{\text{Vol}(Y)}.$$

It follows that there is a strictly positive function  $\bar{\rho}_X$  on  $X$  with  $\bar{\rho}_X \circ \pi = \rho_Y$ , where  $\pi : Y \rightarrow X$  is the projection. Since  $d_X(x, x') = d_Y(\pi^{-1}(x), \pi^{-1}(x'))$ , then  $\bar{\rho}_X$  is Lipschitz.

The projection  $\pi_i : FM_i \rightarrow M_i$  is a Riemannian submersion with totally geodesic fibers isometric to  $O(n)$ . Thus,

$$d\tilde{\mu}_i = \frac{dV_{FM_i}}{\text{Vol}(O(n)) \times \text{Vol}(M_i)}.$$

Then it follows that  $(\pi_i)_*(\tilde{\mu}_i) = \mu_i$ . For any continuous function  $f$  on  $X$ , we have by Fubini's theorem

$$\begin{aligned} \int_X f(x) \rho_X(x) \frac{dV_h}{\text{Vol}(X)} &= \lim_{i \rightarrow \infty} \int_{M_i} (f \circ f_i) d\mu_{M_i} \\ &= \lim_{i \rightarrow \infty} \int_{FM_i} (f \circ f_i \circ \pi_i) d\tilde{\mu}_i = \lim_{i \rightarrow \infty} \int_{FM_i} (f \circ \pi \circ \tilde{f}_i) d\tilde{\mu}_i \\ &= \int_Y (f \circ \pi) d\tilde{\mu} = \int_Y (f \circ \pi) \rho_Y \frac{dV_Y}{\text{Vol}(Y)} \\ &= \frac{1}{\text{Vol}(Y)} \int_X \left( \int_{\pi^{-1}(x)} \rho_Y(y) d\mathcal{H}_{\pi^{-1}(x)}^\ell(y) \right) f(x) dV_X(x) \\ &= \int_X f(x) \bar{\rho}_X(x) \text{Vol}_\ell(\pi^{-1}(x)) \frac{\text{Vol}(X)}{\text{Vol}(Y)} d\mu_X, \end{aligned}$$

where  $\mathcal{H}_{\pi^{-1}(x)}^\ell$  is the  $\ell$ -dimensional Hausdorff measure of  $\pi^{-1}(x)$  with  $\ell = \dim O(n)$ . It follows

$$\pi_*(\tilde{\mu}) = \mu \tag{40}$$



and the function

$$\rho_X(x) = \bar{\rho}_X(x) \text{Vol}_\ell(\pi^{-1}(x)) \frac{\text{Vol}(X)}{\text{Vol}(Y)} \quad (41)$$

is the required density function with  $d\mu_X = \rho_X dV_h / \text{Vol}(V)$ . Since  $\pi^{-1}(x) \simeq O(n)/H_{\bar{x}}$  and  $H_{\bar{x}} \simeq G_x$  (see Theorem 2.5), we have  $\text{Vol}_\ell(\pi^{-1}(x)) = 0$  if and only if  $\dim G_x > 0$ , that is,  $x$  is not an orbifold point of  $X$ .

In the future, we will need also the properties of the the class of the orthonormal framebundles  $FM$  over Riemannian manifolds  $(M, p, \mu) \in \mathfrak{MM}_p(n, \Lambda, D)$ .  $FM$  are equipped with the Riemannian metric,  $\tilde{h}$ , inherited from  $(M, h)$  and the corresponding probability measure,  $\tilde{\mu}$ , see (39). We denote this class by  $\mathfrak{FM}(n, \Lambda, D)$ . Clearly,

$$\begin{aligned} n_F &= \dim(FM) = n + \dim(O(n)), \quad \text{diam}(FM) \leq D_F, \\ |R(FM, h_F)| &\leq \Lambda_F^2, \quad \text{for } FM \in \mathfrak{FM}(n, \Lambda, D). \end{aligned} \quad (42)$$

The closure,  $\overline{\mathfrak{FM}}(n, \Lambda, D)$  with respect to the measured GH topology gives rise to the  $C_*^3$  manifolds  $Y$ , equipped with  $C_*^2$ -smooth Riemannian metric  $\tilde{h}$ , which appear in Theorem 2.3, (2), and appear in the proof of Lemma 3.1 when dealing with non-smooth  $X$ . Analysing the proof of Lemma 3.1, we have

**Corollary 3.4** *Let  $Y \in \overline{\mathfrak{FM}}$ . Then  $Y$  is  $C_*^3$ -smooth Riemannian manifold with  $C_*^2$ -smooth Riemannian metric  $h_Y$  and strictly positive density function  $\rho_Y \in C_*^2(Y)$ . Moreover,  $O(n)$  acts by isometries on  $Y$  and  $\rho_Y$  is  $O(n)$ -invariant. There is a constant  $\Lambda_F = \Lambda_F(n, \Lambda, D) > 1$  such that*

$$|R(Y)| \leq \Lambda_F^2, \quad Y \in \overline{\mathfrak{FM}}. \quad (43)$$

## 4 On properties of eigenfunctions

In this section we consider some properties of the eigenfunctions  $\phi_j$ ,  $j = 0, 1, \dots$ , of the weighted Laplace operator,  $\Delta_X$ , for  $(X, p, \mu) \in \overline{\mathfrak{MM}}_p$ . Note that the basic spectral properties, including its rigorous definition, of this operator are given in appendix C. Simultaneously, we will consider those

properties for weighted Laplacian on  $(Y, d\mu_Y) \in \overline{\mathfrak{FMM}}_p$ , in which case we denote the eigenvalues and corresponding eigenfunctions by  $\tilde{\lambda}_j, \tilde{\phi}_j, j = 0, 1, \dots$ . As  $h_Y, \rho_Y \in C_*^2(Y)$ , it is natural to consider  $C_*^3$ -smooth transition functions  $\psi_{jk} : \pi_j^{-1}(\tilde{U}_j \cap \tilde{U}_k) \rightarrow \pi_k^{-1}(\tilde{U}_j \cap \tilde{U}_k)$  between coordinate charts of  $(\tilde{U}_j, \tilde{\pi}_j)$  and  $(\tilde{U}_k, \tilde{\pi}_k)$  on  $Y$ . In these coordinates we can invariantly define the Sobolev spaces  $W^{k,q}(Y)$ ,  $k \in \{0, 1, 2, 3\}$ ,  $1 \leq p < \infty$ , the Holder classes  $C^{l,\alpha}(Y)$ ,  $l \in \{0, 1, 2\}$ ,  $0 \leq \alpha < 1$  and the Zygmund space  $C_*^3(Y)$ .

Analogously, due to Theorem 3.3 with  $M = Y$ ,  $G = O(n)$ , we can also define spaces  $W_O^{k,q}(Y) \subset W^{k,q}(Y)$ ,  $C_O^{l,\alpha}(Y) \subset C^{l,\alpha}(Y)$  and  $C_{*,O}^3(Y) \subset C_*^3(Y)$  which consist of functions invariant with respect to the action of  $O(n)$ .

Denote by  $\mathbb{P}_O$  the projector

$$\mathbb{P}_O : L^2(Y) \rightarrow L_O^2(Y), \quad (\mathbb{P}_O u^*)(y) = \int_{O(n)} u^*(o(y)) d\mu_o, \quad (44)$$

where  $d\mu_o$  is the normalized Haar measure on  $O(n)$ .

**Lemma 4.1** *For any  $k \in \{0, 1, 2, 3\}$ ,  $l \in \{0, 1, 2\}$ ,  $1 \leq q < \infty$  and  $0 \leq \alpha < 1$ ,*

$$\begin{aligned} \mathbb{P}_O : W^{k,q}(Y) &\rightarrow W_O^{k,q}(Y), \quad \mathbb{P}_O : C^{l,\alpha}(Y) \rightarrow C_O^{l,\alpha}(Y), \\ \mathbb{P}_O : C_*^3(Y) &\subset C_{*,O}^3(Y), \end{aligned} \quad (45)$$

*are bounded projectors. Moreover,*

$$\mathbb{P}_O : L^2(Y, \mu_Y) \rightarrow L_O^2(Y, \mu_Y),$$

*is an orthogonal projector.*

*In addition, for any  $1 \leq q < \infty$ ,  $W_O^{k,q}(Y)$ ,  $k \in \{0, 1, 2, 3\}$  and  $C_{*,O}^3(Y)$  are dense in  $L_O^q(Y, \mu_Y)$ .*

**Proof.** Since, by Theorem 3.3,  $O(n)$  acts by  $C_*^3$ -isometries which also preserve the measure  $\mu_Y$ , then, for any  $o \in O(n)$ ,

$$\|o^* u^*\|_{W^{k,q}(Y)} = \|u^*\|_{W^{k,q}(Y)}, \quad (o^* u^*)(y) = u^*(o(y)),$$

and the similar equation is valid for  $C^{l,\alpha}$  and  $C_*^3$ -norms. Therefore, by integral Minkowski inequality

$$\|\mathbb{P}_O u^*\|_{W^{k,q}(Y)} \leq \int_{O(n)} \|o^* u^*\|_{W^{k,q}(Y)} d\mu_o = \|o^* u^*\|_{W^{k,q}(Y)}.$$

Using the definition of the norms in  $C^{l,\alpha}$  and  $C_*^3$  for  $\mathbb{P}_O u$  in these norms.

Next consider  $L^2(Y)$ . Then,

$$\begin{aligned} (\mathbb{P}_O u^*, v^*)_{L^2(Y)} &= \int_Y \left( \int_{O(n)} u^*(o(y)) d\mu_o \right) v^*(y) d\mu_Y(y) \\ &= \int_{O(n)} \left( \int_Y u^*(o(y)) v^*(y) d\mu_Y(y) \right) d\mu_o \\ &= \int_{O(n)} \left( \int_Y u^*(\tilde{y}) v^*(o^{-1}(\tilde{y})) d\mu_Y(\tilde{y}) \right) d\mu_o, \end{aligned}$$

where, at the last step we make the substitution  $\tilde{y} = oy$  and use the invariance of  $\mu_o$  and the fact that  $O(n)$  acts by isometries. Thus,

$$(\mathbb{P}_O u^*, v^*) = \int_Y u^*(y) \left( \int_{O(n)} v^*(o(y)) d\mu_o \right) d\mu_Y = (u^*, \mathbb{P}_O v^*).$$

To obtain the density of  $C_{*,O}^3$  and, therefore,  $W_{O,O}^{k,q}$  in  $L_O^q$ , we first approximate  $u^* \in L_O^q(Y)$  by  $C_*^3$ -functions  $\tilde{u}_k^*$  and then consider

$$u_k^* = \mathbb{P}_O \tilde{u}_k^* \in C_{*,O}^3(Y),$$

To complete the proof, let us show that

$$\|u^* - u_k^*\|_{L^q(Y)} \leq \|u^* - \tilde{u}_k^*\|_{L^q(Y)}. \quad (46)$$

Since  $u^*$  is  $O(n)$ -invariant,

$$u^*(y) = \int_{O(n)} u^*(o(y)) d\mu_o, \quad y \in Y.$$

Therefore,

$$u^*(y) - u_k^*(y) = \int_{O(n)} (u^* - \tilde{u}_k^*)(o(y)) d\mu_o,$$

and (46) follows from the integral Minkowski inequality.

QED

Analogously, as  $X^{reg}$  is a  $C_*^3$ -smooth manifold, we can define function spaces  $W_{loc}^{k,q}(X^{reg})$ ,  $C^{l,\alpha}(X^{reg})$  and  $C_*^3(X^{reg})$ . In addition, as  $X$  is a metric space, we can define the space of Lipschitz function,  $C^{0,1}(X)$ .

**Lemma 4.2** (1). Let  $(Y_p, \mu_Y) \in \overline{\mathfrak{FMM}}_p$ . Then,  $\tilde{\phi}_j \in C_*^3(Y)$ .

(2). Let  $(X, p, \mu_X) \in \overline{\mathfrak{MM}}_p$ . Then,

$$\phi_j \in C^{0,1}(X) \cap C_*^3(X^{reg}) \cap \left( \bigcap_{1 \leq q < \infty} W_{loc}^{3,q}(X^{reg}) \right). \quad (47)$$

**Proof.** (1), In local coordinates on  $Y$ ,

$$-\frac{1}{\sqrt{h_Y} \rho_Y} \partial_i (\sqrt{h_Y} h_Y^{il} \rho_Y \partial_l \tilde{\phi}_j) = \lambda_j \tilde{\phi}_j.$$

We use the interpolation arguments for the interior Schauder estimates, see e.g. [30, Th. 6.17]. As  $h_Y$  and  $\rho_Y$  are  $C_*^2$ -smooth, an extension of these arguments to Zygmund spaces shows, cf. [2, Prop. 2.4.1], also [61, Thm. 14.4.2-3] or [64] for the domains in  $\mathbb{R}^n$ , that  $\phi_j \in C_*^3(Y)$ . As the embeddings  $\mathcal{I} : C^{k,\alpha}(Y) = B_{\infty,\infty}^{k+\alpha}(Y) \rightarrow W^{k+\alpha,q}(Y)$ , for  $n \in \mathbb{Z}_+$ ,  $0 < \alpha < 1$ , and  $1 \leq q < \infty$ , are continuous [8, Thm. 6.4.4], we see, using interpolation, that  $\mathcal{I} : C_*^3(Y) \rightarrow W^{3,p}(Y)$ ,  $p < \infty$  is continuous.

(2). Let  $(X, \mu_X) = (Y, \mu_Y)/O(n)$ . Denote by  $\Delta_Y^O$  the invariant part of  $\Delta_Y$  on  $L_O^2(Y)$ , where  $L_O^2(Y)$  is the subspace of  $L^2(Y, \mu_Y)$  of  $O(n)$ -invariant functions,

$$L^2(Y, \mu_Y) = L_O^2(Y) \oplus L_\perp^2(Y),$$

see Appendix C for details. By [24, Lemma (7.1)], see also Appendix C,

$$\text{spec}(\Delta_Y^O) = \text{spec}(\Delta_X), \quad \phi_j^O = \pi^* \phi_j, \quad (48)$$

where  $\lambda_j^O, \phi_j^O$  stand for the eigenvalues and eigenfunctions of  $\Delta_Y^O$ . Let, for  $x, y \in X$ ,  $x^* \in \pi^{-1}(x)$ ,  $y^* \in \pi^{-1}(y)$ , satisfy

$$d_X(x, y) = d_Y(\pi^{-1}(x), \pi^{-1}(y)) = d_Y(x^*, y^*).$$

Then,

$$|\phi_j(x) - \phi_j(y)| = |\tilde{\phi}_j^O(x^*) - \tilde{\phi}_j^O(y^*)| \leq \|\tilde{\phi}_j^O\|_{C^{0,1}(Y)} d_X(x, y), \quad (49)$$

which proves the first inclusion in (47).

Next consider  $Y^r = \pi^{-1}(X^{reg})$ , so that  $X^{reg} = Y^r/O(n)$ . Note that  $Y^r$  is invariant with respect to  $O(n)$  which acts by the measure preserving isometries, see Theorem 3.3. In addition, by Theorem 2.5 (4),  $O(n)$  acts freely on  $Y^r$ . Thus, the remaining inclusions in (47) follow from (48).

QED

Denote by  $Z_X, Z_Y, Z_Y^O$  the linear subspaces of finite linear combinations of the eigenfunctions  $\{\phi_j\}_{j=0}^\infty, \{\tilde{\phi}_j\}_{j=0}^\infty$ , and  $\{\phi_j^O\}_{j=0}^\infty$ , correspondingly.

**Proposition 4.3** 1.  $Z_Y, Z_Y^O$  are dense in  $W^{k,q}(Y), W_O^{3,q}(Y)$ , for any  $k \in \{0, 1, 2, 3\}, 1 \leq q < \infty$  and also  $C^{l,\alpha}(Y), C_O^{l,\alpha}(Y), l \in \{0, 1, 2\}, 0 \leq \alpha < 1$  and  $C_*^3(Y), C_{*,O}^3(Y)$ , correspondingly.

2.  $Z_X$  is dense in  $W_{loc}^{k,q}(X^{reg})$  and also  $C^{l,\alpha}(X^{reg})$  and  $C_*^3(X^{reg})$ . In addition,  $Z_X$  is dense in  $C^{0,1}(X)$ .

**Proof.** 1. Let  $\Delta_q$  be the weighted Laplacian  $\Delta_Y$  with the domain of definition  $\mathcal{D}(\Delta_q) = W^{2,q}(Y), 1 \leq q < \infty$  so that  $\Delta_q : W^{2,q}(Y) \rightarrow L^q(Y)$ . Note that, for  $2 < q < \infty$ ,  $\Delta_q$  is the restriction of  $-A := \Delta_2 = \Delta_Y : W^{2,2}(Y) \rightarrow L^2(Y)$  and, for  $1 < q < 2$ ,  $\Delta_q$  is the continuous extension of  $\Delta_2$ . We start with the case  $q = 2$ , when

$$(I + A)^{-1} : L^2(Y) \rightarrow W^{2,2}(Y)$$

is an isomorphism.

If  $f \in L^q(Y)$  with  $2 \leq q < \infty$ , we see using [45, Thm. 9.4.1] that  $u = (I + A)^{-1} f \in W^{2,q}(Y)$ . Moreover, by [2, Prop. 2.4.1] and [30, Thm. 9.19, 9.30], if  $f \in C_*^0(Y)$ , then  $u \in C_*^2(Y)$ . By the Closed graph theorem, the above yield that, for  $2 \leq q < \infty$ , that the operators

$$(I + A)^{-1} : L^q(Y) \rightarrow W^{2,q}(Y), \quad (I + A)^{-1} : C_*^0(Y) \rightarrow C_*^2(Y) \quad (50)$$

are bounded and surjective and, therefore, are isomorphisms.

Let us next show that  $Z_Y$  is dense in  $L^q(Y)$  for all  $2 \leq q < \infty$ . As  $\tilde{\phi}_j, j \in \mathbb{Z}_+$ , form an orthonormal basis of  $L^2(Y)$ , the space  $Z_Y$  is dense in  $L^q(Y)$  with  $q = 2$ .

Let  $q > 2, m \in \mathbb{Z}_+$  and  $0 < a < 2n_Y^{-1}$ , where  $n_Y = \dim(Y)$ , be such that  $ma = 1/2 - 1/q$ . We define  $q(k) \geq 2, k = 0, 1, \dots, m$ , by setting

$$q(0) = q, \quad \frac{1}{q(k)} = \frac{1}{q(k-1)} + a, \quad k = 1, 2, \dots, m,$$

so that then  $q(m) = 2$ .

Recall that we have already shown that  $Z_Y \subset L^{q(m)}(Y)$  is dense. Assume next that we have shown that  $Z_Y \subset L^{q(k)}(Y)$  is dense for some  $k = 1, 2, \dots, m$ . Then, as  $I + A : Z_Y \rightarrow Z_Y$  is bijective and  $(I + A)^{-1} : L^{q(k)}(Y) \rightarrow W^{2,q(k)}(Y)$  is an isomorphism, we see that  $Z_Y \subset W^{2,q(k)}(Y)$  is dense. By Sobolev embedding, for  $\frac{1}{q'} = \frac{1}{q(k)} + \frac{2}{n(Y)}$ , so that  $q' > q(k-1)$ , we have  $W^{2,q(k)}(Y) \subset L^{q'}(Y) \subset L^{q(k-1)}(Y)$ . By using the fact that  $C_*^3(Y)$  is dense both in  $W^{2,q(k)}(Y)$  and  $L^{q(k-1)}(Y)$ , we see that  $W^{2,q(k)}(Y) \subset L^{q(k-1)}(Y)$  is dense. Since  $Z_Y \subset W^{2,q(k)}(Y)$  is dense, this implies that  $Z_Y \subset L^{q(k-1)}(Y)$  is dense. Iterating this argument  $m$  times, we see that  $Z_Y$  is dense in  $L^{q(0)}(Y) = L^q(Y)$ .

As noted above, for  $2 \leq q < \infty$  the operator  $(I + A)^{-1} : L^q(Y) \rightarrow W^{2,q}(Y)$  is isomorphism, and as  $I + A : Z_Y \rightarrow Z_Y$  is bijective the above yields that  $Z_Y$  is dense in  $W^{2,q}(Y)$ .

To prove the density of  $Z_Y$  in  $W^{3,q}(Y)$ , we use the fact that, similar to the above,

$$(I + A)^{-1} : W^{1,q}(Y) \rightarrow W^{3,q}(Y), \quad 2 \leq q < \infty,$$

is an isomorphism. As we already know that  $Z_Y$  is dense in  $W^{1,q}(Y)$ , this implies that  $Z_Y$  is dense in  $W^{3,q}(Y)$ ,  $g \geq 2$ . At last, using the density of  $W^{k,q}(Y)$  in  $W^{k,q'}(Y)$ ,  $q' \leq q$ , we extend  $q$  to  $[1, \infty)$ .

Let  $q > n_Y/2$ . Since  $W^{3,q}(Y) \subset C^{0,1}(Y)$ , the set  $Z_Y$  is dense in  $C^{0,1}(Y)$ . Since  $-\Delta_Y$  on  $C_*^3(Y)$  is the restriction of  $A$ , by extending classical interior elliptic regularity results from  $C^{k,\alpha}$ -classes, see e.g. [30, Th. 6.17], to Zygmund classes using the same construction as in e.g. [2, Prop. 2.4.1], see also [61, Thm. 14.4.2-3] or [64], we see that

$$(I + A)^{-1} : C^{0,1}(Y) \rightarrow C_*^3(Y)$$

is bounded. Using again that  $(I + A)^{-1} Z_Y = Z_Y$ , this implies that  $Z_Y$  is dense in  $C_*^3(Y)$ .

Turning to the case of  $Z_Y^O$ , we just note that

$$\begin{aligned} (I - \Delta_Y)^{-1} : L_O^q(Y) &\rightarrow W_O^{2,q}(Y), & (I - \Delta_Y)^{-1} : W_O^{1,q}(Y) &\rightarrow W_O^{3,q}(Y), \\ (I - \Delta_Y)^{-1} : C_{*,O}^1(Y) &\rightarrow C_{*,O}^3(Y), \end{aligned}$$

are isomorphisms. Taking into the account that  $Z_Y^O$  is dense in  $L_O^2(Y)$ , repeating the above arguments brings about the desired result.

2. Note that

$$C_O^{0,1}(Y) = \pi^*(C^{0,1}(X)), \quad (51)$$

see also Appendix C, (240) or [24, Sec. 7] where the classes  $C^1$  can be easily substituted by classes  $C^{0,1}$ . This, together with (49), (48) and the density of  $Z_Y^O$  in  $C^{(0,1)}(Y)$  provides the density of  $Z_X$  in  $C^{0,1}(X)$ .

Next, to prove the density of  $Z_X|_{X^{reg}}$  in  $W_{loc}^{k,q}(X^{reg})$ ,  $k \in \{0, 1, 2, 3\}$ ,  $1 \leq q < \infty$ , or in  $C^{l,\alpha}(X^{reg})$ ,  $l \in \{0, 1, 2\}$ ,  $0 \leq \alpha < 1$ , it is sufficient to show that  $Z_X|_{X^{reg}}$  is dense in  $C_*^3(X^{reg})$ .

Let  $K$  be a compact in  $X^{reg}$ . To prove that last statement, it is sufficient to show that any  $u \in C_*^3(X^{reg})$ ,  $\text{supp}(u) \subset K$ , can be approximated, in  $C_*^3(K)$ , by a sequence of functions  $z_m|_K$ ,  $z_m \in Z_X$ . To this end, consider

$$u^* = \pi^*(u) \in C_{*,O}^3(Y).$$

Since  $O(n)$  acts freely on  $\pi^{-1}(K)$ , it follows from Theorem 3.3 that *visa versa*

$$\pi_* : C_{*,O}^3(\pi^{-1}(K)) \rightarrow C_*^3(K) \quad (52)$$

is a bounded operator. Note that to see this, it is sufficient to use, near an arbitrary  $y_0 \in \pi^{-1}(K)$ , the local coordinates  $(o, x)$ . Here  $x$  are local coordinates near  $x_0 = \pi(y_0)$  and  $o$  are local coordinates near  $o_0$ , where  $y_0 = o_0(s(x_0))$ , cf. (35) with  $g$  replaced by  $o$  and  $x$  being local coordinates in  $X^{reg}$ .

By part (1), there is a sequence  $\{z_m^O\}_{m=1}^\infty$ ,  $z_m^O \in Z_Y^O$  which approximates  $u^*$  in  $C_*^3(Y)$  as  $m \rightarrow \infty$ . Thus,  $\{z_m^O|_{\pi^{-1}(K)}\}_{m=1}^\infty$  approximates  $u^*|_{\pi^{-1}(K)}$  in  $C_{*,O}^3(\pi^{-1}(K))$  and, by (52),

$$\lim_{m \rightarrow \infty} \pi_*(z_m^O)|_K = \pi_*(u^*)|_K = u|_K \quad \text{in } C_*^3(K).$$

However, by (48),  $\pi_*(z_m^O) \in Z_X$ .

QED

**Corollary 4.4** (i) *The map  $\Phi : X \rightarrow \mathbb{R}^{\mathbb{Z}_+}$  defined by  $\Phi(x) = (\phi_j(x))_{j=1}^\infty$  is one-to-one, that is, if  $x \neq y$ , there is an index  $j \in \mathbb{Z}_+$  such that*

$$\phi_j(x) \neq \phi_j(y).$$

(ii) For any  $x_0 \in X^{reg}$  there a neighbourhood  $U \subset X^{reg}$  of  $x_0$  and indeces  $\mathbf{j} = (j_1, \dots, j_d)$ , where  $d = \dim(X)$ , such that

$$\Phi_{\mathbf{j}} : U \rightarrow \mathbb{R}^d, \quad \Phi_{\mathbf{j}}(x) = (\phi_{j_k}(x))_{k=1}^d, \quad x \in U,$$

are  $C_*^3$ -smooth coordinates in  $U$ .

**Proof.** (i) Assume the opposite, that is, there are  $x, y \in X$ ,  $x \neq y$  for which

$$\phi_j(x) = \phi_j(y), \quad \text{for all } j = 0, 1, \dots \quad (53)$$

Let  $v \in C^{0,1}(X)$  with  $v(x) = 1$ ,  $v(y) = 0$ , e.g.  $v(x') = d_X(y, x')/d_X(x, y)$ ,  $x' \in X$ . By Proposition 4.3 2., there are  $z_m \in Z_X$ ,  $m = 1, 2, \dots$ , such that  $z_m \rightarrow v$  in  $C^{0,1}(X)$  as  $m \rightarrow \infty$ . In particular,  $z_m(x) \rightarrow v(x) = 1$  and  $z_m(y) \rightarrow v(y) = 0$ . However, if (53) is valid then  $z_m(x) = z_m(y)$  for all  $m$  and thus,  $v(x) = v(y)$ . This is a contradiction and thus (i) has to hold.

(ii) We need only to show that there is  $\mathbf{j} = \mathbf{j}(x_0)$  such that  $(d\phi_{j_1}(x_0), \dots, d\phi_{j_d}(x_0))$ , are linearly independent.

Let  $(x^1, \dots, x^d)$  be  $C_*^3$ -smooth local coordinates in a neighborhood  $V \subset X^{reg}$  of  $x_0$ . Let  $\chi \in C_*^3(X^{reg})$  be supported in  $V$  such that  $\chi(x) = 1$  in some neighborhood of  $x_0$  and let  $\widehat{\chi}_i(x) = x^i \chi(x)$ ,  $i = 1, 2, \dots, d$ . By Proposition 4.3 2., there are  $u_m^i \in Z$ ,  $m = 1, 2, \dots$ , such that  $d(u_m^i) \rightarrow d\widehat{\chi}_i$  in  $\mathbf{D}_*^2(X)$ , where  $\mathbf{D}_*^2(X^{reg})$  is the space of  $C_*^2$ -smooth differential 1-forms on  $X^{reg}$ . Since  $d\widehat{\chi}_i = dx^i$  near  $x_0$ , this yields that, for some  $m$ ,  $du_m^i(x_0)$ ,  $i = 1, \dots, d$ , are linearly independent. This implies the result.

QED

To understand which choice of  $\mathbf{j}$  is appropriate for coordinates near  $x_0$ , we can use the following lemma.

**Lemma 4.5** *Let  $X \in \overline{\mathfrak{MM}}_p$  and  $x_0 \in X^{reg}$ . Assume that, for  $\mathbf{i} := (i_1, \dots, i_d) \in \mathbb{Z}_+^d$ , there is a neighborhood  $W_{\mathbf{i}} \subset \mathbb{R}^d$ ,  $\Phi_{\mathbf{i}}(x_0) \in W_{\mathbf{i}}$ , such that, for any  $\ell \in \mathbb{Z}_+^d$ , the function  $\phi_{\ell} \circ \Phi_{\mathbf{i}}^{-1} : W_{\mathbf{i}} \rightarrow \mathbb{R}$  is  $C_*^3$ -smooth. Then there is a neighborhood  $U$  of  $x_0$  such that  $\Phi_{\mathbf{i}} : U \rightarrow \Phi_{\mathbf{i}}(U) \subset \mathbb{R}^d$  is a  $C_*^3$ -smooth diffeomorphism. Thus,  $(U, \Phi_{\mathbf{i}})$  forms a local coordinate chart in  $X^{reg}$ .*

**Proof.** By Corollary 4.4 there is  $\mathbf{j} := (j_1, \dots, j_d) \in \mathbb{Z}_+^d$  such that  $(\phi_{j_1}, \dots, \phi_{j_d})$  are  $C_*^3$ -smooth coordinates near  $\Phi_{\mathbf{j}}(x_0)$ . Thus, for any  $\ell$ ,  $\phi_{\ell} \circ \Phi_{\mathbf{j}}^{-1}$  is  $C_*^3$ -smooth



in some neighbourhood  $W_j \subset \mathbb{R}^n$ ,  $\Phi_j(x_0) \in W_j$ . On the other hand, by our assumption, the map  $H = \Phi_j \circ \Phi_i^{-1}$  is also  $C_*^3$ -smooth in some neighborhood  $W_i \subset \mathbb{R}^n$  of  $\Phi_i(x_0)$ . Denote by  $H'$  the map  $H' = \Phi_i \circ \Phi_j^{-1}$ ,  $H' : W_j \rightarrow \mathbb{R}^n$ . Then, on a possibly smaller neighbourhood  $V' \subset W_j$ ,  $\Phi_j(x_0) \in V'$ , we have  $H \circ H' = Id$ . Therefore, in particular,

$$DH(\Phi_i(x_0)) \circ DH'(\Phi_j(x_0)) = Id.$$

This implies that  $DH(\Phi_i(x_0))$  is invertible so that  $\Phi_i$  is a  $C_*^3$ -smooth coordinate map near  $x_0$ .

QED

The above lemmas show that, given the eigenfunctions  $\{\phi_j(x)\}_{j=0}^\infty$  on and open set  $\Omega^{reg} \subset X^{reg}$ ,  $X \in \overline{\mathfrak{MM}}_p$ , we can find the topological and  $C_*^3$ -differentiable structure of  $\Omega^{reg} \subset X^{reg}$ . Next we consider the metric tensor  $h$  and the density  $\rho$  on  $\Omega^{reg}$ .

**Lemma 4.6** *The set  $\Omega^{reg}$ , the eigenvalues  $\{\lambda_j\}_{j=0}^\infty$  of  $-\Delta_X$ , and the corresponding eigenfunctions  $\{\phi_j(x)\}_{j=0}^\infty$ ,  $x \in \Omega^{reg}$  determine uniquely the metric  $h|_{\Omega^{reg}}$  and, a weight function  $\dot{\rho}$  in  $\Omega$  which satisfies  $\dot{\rho}(x) = c\rho(x)$ , for  $x \in \Omega^{reg}$ , where  $c > 0$  is some constant.*

**Proof.** Let  $x_0 \in \Omega^{reg}$  and  $\mathbf{x} : U \rightarrow \mathbb{R}^d$  be  $C_*^3$ -smooth coordinates in a neighborhood  $U \subset \Omega^{reg}$  of  $x_0$ ,  $\mathbf{x}(x_0) = 0$ . Next we do computations in the local coordinates  $\mathbf{x}(x) = (x^1, \dots, x^d)$  and identify  $U$  and  $\mathbf{x}(U)$  as well as  $x_0$  and 0. Let  $\chi(x)$  be a  $C_*^3$ -smooth function which is supported in a compact subset of  $U$  so that  $\chi(x) = 1$  in a neighborhood  $V$  of 0. Let

$$\begin{aligned} \chi_i(x) &= x^i \chi(x), \quad i = 1, \dots, d, \\ \chi_{j,k}(x) &= x^j x^k \chi(x), \quad 1 \leq j \leq k \leq d. \end{aligned} \tag{54}$$

Let now  $y \in V$ . We know that there are  $z_i^\ell, z_{j,k}^\ell \in Z_X$ , such that, in  $C_*^3(U)$ ,

$$\chi_i = \lim_{\ell \rightarrow \infty} z_i^\ell, \quad \chi_{j,k} = \lim_{\ell \rightarrow \infty} z_{j,k}^\ell.$$

This implies that there is  $N \in \mathbb{Z}_+$ , such that the vectors  $\Psi_y(\phi_p)$ ,  $p = 0, 1, \dots, N$ , span the space  $\mathbb{R}^{d(d+3)/2}$ , where

$$\Psi_y[f] = \left( \left( \frac{\partial f}{\partial x^i}(y) \right)_{i=1}^d, \left( \frac{\partial^2 f}{\partial x^j \partial x^k}(y) \right)_{(j,k) \in A} \right),$$

where  $A$  is the set of pairs  $(j, k)$  such that  $1 \leq j, k \leq d$  and  $j \leq k$ . Let us consider the equations

$$-\Delta_X \phi_p(y) := -h^{jk}(y) \frac{\partial^2}{\partial x^j \partial x^k} \phi_p(y) - a^l(y) \frac{\partial}{\partial x^j} \phi_p(y) = \lambda_p \phi_p(y), \quad (55)$$

where  $p = 0, 1, \dots, N$ , and

$$a^j = \frac{1}{\sqrt{h} \rho} \partial_k \left( \sqrt{h} h^{jk} \rho \right) = \frac{1}{\sqrt{h}} \partial_k \left( \sqrt{h} h^{jk} \right) + h^{jk} \partial_k \ln(\rho).$$

Solving the system (55), we find the coefficients  $h^{jk}(y)$  and  $a^j(y)$ . Using the equation for  $a^j$  we find  $\nabla(\ln \rho)$  at  $y$ . As  $y \in V$  is arbitrary, this means that we can determine the metric tensor  $h_{jk}$  and the function  $\rho$  (up to a multiplicative constant) in  $V$ . QED

In the future, we will chose and fix some positive solution,  $\widehat{\rho}$ , for the equations for  $a^l$ , so that

$$\int_{\Omega^{reg}} \widehat{\rho}(x) dV_h = 1, \quad \widehat{\rho} = \widehat{c} \rho, \quad (56)$$

with some  $\widehat{c} > 0$ .

At the end of this section, we show that the point heat data determines, via the local spectral data, the metric-measure structure of any open set  $\Omega \subset X$ . Recall that the point heat data for a weighted Laplacian on  $X \in \overline{\mathcal{MM}}_p$ , associated with  $\Omega$  is defined as follows:

Let  $H(x, y, t)$ ,  $x, y \in X$ ,  $t \in \mathbb{R}_+$ , be the heat kernel for the weighted Laplace operator  $\Delta_X$ . Let  $\Omega \subset X$ ,  $\Omega \neq \emptyset$ , be open and  $\{z_i, i \in \mathbb{Z}_+\}$  be dense in  $\Omega$ . Let also  $\{t_l; l \in \mathbb{Z}_+\}$  be dense in  $\mathbb{R}_+$ . Then the point heat data (PHD) is the set

$$\text{PHD} = (H(z_i, z_k; t_l))_{i,k,l \in \mathbb{Z}_+}. \quad (57)$$

**Lemma 4.7** *Let the set  $\text{PHD} = (H(z_i, z_k; t_l))_{i,k,l \in \mathbb{Z}_+}$  be point heat data associated with the weighted Laplacian  $\Delta_X$  on some open  $\Omega \subset X$ . Then these PHD uniquely determine  $\Omega$ ,  $\Omega^{reg} = \Omega \cap X^{reg}$ , the metric  $h|_{\Omega^{reg}}$ , and the eigenvalues  $\lambda_j$ . They also determine, up to an orthogonal transformation in  $\mathcal{L}(\lambda_j)$ , the orthonormal eigenfunctions,  $\phi_j|_{\Omega^{reg}}$ ,  $j \in \mathbb{Z}_+$ , and the density  $\widehat{\rho}$  in  $\Omega^{reg}$  which satisfies  $\widehat{\rho}(x) = c' \rho(x)$  for  $x \in \Omega$ , where  $c' \in \mathbb{R}_+$  is a constant so that  $\Omega$  has volume one with respect to the measure  $\widehat{\rho} dV_h$ .*

Here and later we denote by  $\mathcal{L}(\lambda_j)$  the eigenspace of  $-\Delta_X$ , corresponding to the (multiple) eigenvalue  $\lambda_j$ .

**Proof.** Observe first that, due to the analiticity of  $H(x, y; t)$  with respect to  $t$ ,  $\text{Re}(t) > 0$ ,  $H_{i,k;l}$ ,  $l \in \mathbb{Z}_+$  determine the function  $H_{ik}(t) \in C(0, \infty)$ ,

$$H_{ik}(t) = H(z_i, z_k; t).$$

Take the Laplace trasform of  $H_{ik}(t)$ ,

$$H_{ik}(t) \mapsto \widehat{H}_{ik}(\omega) = \int_0^\infty e^{-t\omega} H_{ik}(t) dt, \quad \text{Re}(\omega) > 0, \quad (58)$$

and denote  $\phi_{j;i} = \phi_j(z_i)$ . Note that, as follows from the Weyl's asymptotics for eigenfunctions together with Sobolev's embedding theorem, for a given  $(X, p, \mu)$  we have

$$\lambda_j \geq C^{-1} j^{2/n}, \quad \|\phi_j\|_{C^{0,1}(X)} \leq C j^{\alpha(n)}, \quad (59)$$

cf. Proposition 5.2 and Lemma 5.6. This implies that, in the standard representation,

$$H(x, y, t) = \sum_{j=1}^\infty e^{-\lambda_j t} \phi_j(x) \phi_j(y), \quad (60)$$

the series in the right-hand side convergesw in  $C^{0,1}(X \times X \times (0, \infty))$ . Thus the integral (58) converges absolutely for  $\text{Re}(\omega) > 0$  and has a meromorphic continuation to the whole plane  $\mathbb{C}$ . Indeed, let

$$H_{ik}^j(t) = H_{ik}(t) - \sum_{j'=1}^{j-1} e^{-\lambda_{j'} t} \phi_{j';i} \phi_{j';k}.$$

Then, using (59),

$$|H_{ik}^j(t)| \leq C_\varepsilon e^{(-[\lambda_j - \varepsilon]t)} (1 + j)^{\beta(n)}, \quad \text{for any } \varepsilon > 0.$$

This, by the same estimate, implies that  $H_{ik}^j(t)$  is analytic for  $\text{Re}(\omega) > -\lambda_j$ . These arguments show that  $\widehat{H}_{ik}(\omega) = \widehat{H}_{ik}^j(\omega) + \sum_{j'=1}^{j-1} (\omega + \lambda_j)^{-1} \phi_{j';i} \phi_{j';k}$  has a meromorphic continuation onto  $\mathbb{C}$  with simple poles at  $\omega = -\lambda_j$ ,  $j \in \mathbb{Z}_+$  and the corresponding residues,

$$\text{Res}_{\omega=-\lambda_j} (\widehat{H}_{ik}(\omega)) = \sum_{\lambda_{j'}=\lambda_j} \phi_{j';i} \phi_{j';k}. \quad (61)$$

As  $\{z_k\}$ ,  $k \in \mathbb{Z}_+$  is dense in  $\Omega$ , we see that the poles of the Laplace transform of  $H_{ik}(t)$ ,  $i, k \in \mathbb{Z}_+$ ,  $t \in (0, \infty)$  determine uniquely  $\lambda_j$  and the residues of the Laplace transform given in (61). Using the results in [41, Lem. 4.9], it follows that those residues determine, up to an orthogonal linear transformation in the group  $O(m_j)$ , where  $m_j$  is the multiplicity of the eigenvalue  $\lambda_j$ , the values  $\phi_{j;i}$ .

Observe now that the map,  $\Phi : \overline{\Omega} \rightarrow \mathbb{R}^{\mathbb{Z}_+}$  is injective and continuous with respect to the product topology in  $\mathbb{R}^{\mathbb{Z}_+}$ . As  $\overline{\Omega}$  is compact,  $\Phi$  is a homeomorphism between  $\overline{\Omega}$  and  $\Phi(\overline{\Omega}) \subset \mathbb{R}^{\mathbb{Z}_+}$ . Thus,  $\phi_{j;i}$ ,  $i, j \in \mathbb{Z}_+$ , determine  $\Phi(\overline{\Omega})$  and, therefore,  $\Phi(\Omega) = (\Phi(\overline{\Omega}))^{int}$ .

We next use Corollary 4.4(2) and Lemmata 4.5 and 4.6 to identify the dense open subset  $\Omega^{reg} = \Omega \cap X^{reg}$ . This is the set where there are  $d$  eigenfunctions  $\Phi_i$  forming  $C_*^3$ -coordinate system in the sense of Corollary 4.4(2) and Lemma 4.5, and  $d(d+3)/2$  eigenfunctions  $\phi_j$ ,  $j = 1, \dots, d(d+3)/2$ , to identify, using Lemma 4.6,  $h$  and  $\hat{\rho}$ , requiring them to belong to  $C_*^2$ .

QED

In the future we need the following result which is a consequence of the proof of Lemma 4.7.

**Corollary 4.8** *For any  $X \in \overline{\mathfrak{MM}}_p$ , assume that  $H_X(x, z, t) = H_X(x', z, t)$ , for  $z \in U$ ,  $t \in (a, b) \subset \mathbb{R}_+$ , where  $U \subset X$  is open. Then  $x = x'$ .*

**Remark 4.9** Analysing the proof of Lemma 4.7 it is clear that, given PHD,  $(H(z_i, z_k; t_l))_{i,k,l \in \mathbb{Z}_+}$ , associated with some open set  $\Omega \in X \in \overline{\mathfrak{MM}}_p$ , we do not need to know *a priori* the dimension  $d = \dim(X)$  of  $X$ . Indeed,  $d$  will be the minimal number so that, for any  $y$  in an open dense set in  $\Omega$  there are  $d$  eigenfunctions,  $\phi_{j(1,y)}, \dots, \phi_{j(d,y)}$ , which form  $C_*^3$ -coordinates in a neighbourhood of  $y$ .

## 5 Continuity of the direct map

Recall that we consider the class  $\mathfrak{MM}_p = \mathfrak{MM}_p(n, \Lambda, D)$  of pointed  $n$ -dimensional Riemannian manifolds  $(M, p_M, \mu_M)$ , equipped with probability measure  $\mu_M$  and with the absolute value of the sectional curvature bounded by  $\Lambda^2 > 0$  and diameters bounded by  $D > 0$ . Recall that the probability

measure  $\mu_M$  is related to the Riemannian metric,  $d\mu_M = dV_h/\text{Vol}(M)$ , where  $dV_h$  is the volume element corresponding to the Riemannian structure on  $M$ . We denote by  $\|\cdot\|_{L^2(M_k)}$  the norm of the space  $L^2(M_k; d\mu_k)$ , that is, the norm is defined using the measure  $\mu_k$ .

In Sections 2, 3 we described the structure of the closure, with respect to the pointed, measured Gromov-Hausdorff topology, of the class  $\mathfrak{MM}_p$ , which we denote by  $\overline{\mathfrak{MM}}_p(n, \Lambda, D)$ . Namely, any  $X \in \overline{\mathfrak{MM}}_p$  is a stratified manifold, see (23) and has the form,  $X = Y/O(n)$ , see Theorem 2.3. Here  $Y$  is  $C_*^2$ -smooth Riemannian manifold with  $C_*^2$ -smooth density function  $\rho_Y$ , see Lemma 3.2, where the group  $O(n)$  acts as  $C_*^3$ -smooth isometries, see Theorem 3.3.

Note that if a sequence  $(M_k, p_k, \mu_k) \in \mathfrak{MM}_p$  converges in the pointed measured GH topology, then the limit is a stratified manifold of dimension  $d < n$ , if and only if  $i_k \rightarrow 0$ , where  $i_k$  is the injectivity radius of  $(M_k, p_k, h_k)$ . Otherwise, the limit is an  $n$ -dimensional  $C_*^2$ -smooth Riemannian manifold.

We associate with any space  $(X, p, \mu_X) \in \overline{\mathfrak{MM}}_p(n, \Lambda, D)$  the weighted Laplace operator,  $\Delta_X$ , and the heat kernel,  $H_X(x, y, t)$ ,  $x, y \in \mathbb{R}_+$ ,  $t > 0$ , such that, for any  $y \in X$ ,

$$(\partial_t^2 - \Delta_X)H_X(\cdot, y, t) = 0, \text{ in } X \times \mathbb{R}_+, \quad H_X(\cdot, y, 0) = \delta_y. \quad (62)$$

Here  $\delta_y$  is the Dirac delta-function with respect to the measure

$$d\mu_X = \rho_X \frac{dV_h}{\text{Vol}(X)}, \quad \mu_X(X) = 1,$$

cf. (3). The goal of this section is to prove the continuity, with respect to the pointed measured Gromov-Hausdorff topology, of the direct map,

$$(X, p_X, \mu_X) \mapsto H_X(x, y, t), \quad x, y \in X, \quad t > 0. \quad (63)$$

Recall that if  $(X, p, \mu_X)$  and  $(X', p', \mu_{X'})$  are  $\varepsilon$ -pmGH close, then there are almost isometries,  $f : X \rightarrow X'$ ,  $f' : X' \rightarrow X$ , such that, for any  $x, y \in X$ ,  $x', y' \in X'$  and any measurable  $S \subset X$ ,  $S' \subset X'$ ,

$$\begin{aligned} |d'(f(x), f(y)) - d(x, y)| &< \varepsilon, \quad |d(f'(x'), f'(y')) - d'(x', y')| < \varepsilon, \\ d'(f(p), p') &< \varepsilon, \quad d(p, f'(p')) < \varepsilon, \\ \mu_X(f^{-1}(S')) &< \mu_{X'}(S^\varepsilon) + \varepsilon, \quad \mu_{X'}(f'^{-1}(S)) < \mu_X(S^\varepsilon) + \varepsilon. \end{aligned} \quad (64)$$

Equations (64) imply, in particular, an existence of  $c\varepsilon$ -nets

$$\{x_i\}_{i=1}^{I(\varepsilon)} \subset X, \{x'_i\}_{i=1}^{I(\varepsilon)} \subset X', \quad \text{with } |d(x_i, x_j) - d'(x'_i, x'_j)| < c\varepsilon. \quad (65)$$

We define the topology in the set of the heat kernels,  $H_X(\cdot, \cdot, t)$ ,  $X \in \overline{\mathfrak{MM}}$ , in the following way

**Definition 5.1** *Let  $H_X(x, y, t)$ ,  $x, y \in X, t > 0$ , and  $H_{X'}(x', y', t)$ ,  $x', y' \in X', t > 0$ , be heat kernels for spaces  $(X, p, \mu_X)$ ,  $(X', p', \mu_{X'}) \in \overline{\mathfrak{MM}}_p$ . Then  $H_X, H_{X'}$  are  $\delta$ -close, if there are  $\delta$ -nets  $\{x_i\}_{i=1}^{I(\delta)}$ ,  $\{x'_i\}_{i=1}^{I(\delta)}$  in  $X, X'$ , correspondingly, which satisfy (65) with  $\delta$  in place of  $c\varepsilon$ , and  $\{t_\ell\}_{\ell=1}^{L(\delta)}$  which is  $\delta$ -dense in  $[\delta, \delta^{-1}]$ , such that*

$$|H_X(x_i, x_j, t_\ell) - H_{X'}(x'_i, x'_j, t_m)| < \delta, \quad 1 \leq i, j \leq I(\delta), \quad 1 \leq \ell \leq L(\delta). \quad (66)$$

## 5.1 Spectral estimates on $\overline{\mathfrak{MM}}$ and $\overline{\mathfrak{JMM}}$

In this subsection, we obtain some estimates for the eigenvalues  $\lambda_p = \lambda_p^X$  and the corresponding eigenfunctions,  $\phi_p = \phi_p^X$ ,  $p \in \mathbb{Z}_+$  of the Laplace operator,  $\Delta_X$ ,

$$-\Delta_X \phi_p = \lambda_p \phi_p,$$

when  $(X, p, \mu_X) \in \overline{\mathfrak{MM}}_p$ .

Our results will be based upon the study of the spectral estimates on  $\overline{\mathfrak{JMM}}$ . Recall that

$$\overline{\mathfrak{JMM}}(n, \Lambda, D) = \overline{\{FM : M \in \mathfrak{M}(n, \Lambda, D)\}},$$

where the closure is taken with respect to the measured GH-topology and  $FM$  is the orthogonal framebundle over  $M$  equipped with the Riemannian metric inherited from  $(M, h)$ , see end of Section 3.

First, we recall the Weyl-type estimate for the eigenvalues. To this end, introduce the counting function,  $N_X(E)$  of  $X$ , which is the number of the eigenvalues of  $-\Delta_X$ , counted with their multiplicities, that are less than  $E$ .

**Proposition 5.2** *1. There exists  $c, \tilde{c}, C > 0$  such that, for any  $X \in \overline{\mathfrak{MM}}_p$ ,*

$$\lambda_p^X \geq c p^{2/n}, \quad p = 0, 1, \dots, \quad N_X(E) \leq 1 + C E^{n/2}, \quad (67)$$

Moreover, denoting  $D_X = \text{diam}(X)$ , we have

$$\lambda_p^X \geq \tilde{c} D_X^{-2} p^{2/n}. \quad (68)$$

2. There exists  $n_F, c_F, C_F > 0$  such that, for any  $Y \in \overline{\mathfrak{FMM}}_p$ ,

$$\lambda_p^Y \geq c_F p^{2/n_F}, \quad p = 0, 1, \dots, \quad N_Y(E) \leq 1 + C_F E^{n_F/2}. \quad (69)$$

**Proof.** By [7], there is  $c > 0$  such that the estimate (67) is valid for all manifolds in  $\mathfrak{MM}$ . In fact, such the estimate is valid even for the class of Riemannian manifolds with Ricci curvature bounded from below and bounded diameter.

Since the pointed measured GH-convergence implies the convergence of individual eigenvalues, see [24], the inequality (67) for eigenvalues remains valid for any  $(X, p, \mu_X) \in \overline{\mathfrak{MM}}_p$ . Since the second inequality follows from the first, this proves (67).

To prove (68), observe that if  $(M, h) \in \mathfrak{MM}$ , then  $(\widetilde{M}, \widetilde{h})$ , which is obtained from  $(M, h)$  by homotety,

$$\widetilde{h} = \frac{D^2}{D_M^2} h,$$

is also in  $\mathfrak{MM}$ . Since

$$\lambda_p(\widetilde{M}) = \frac{D^2}{D_M^2} \lambda_p(M),$$

this proves the result for  $\mathfrak{MM}$ . The case of  $\overline{\mathfrak{MM}}$  is obtained by continuity.

This proves the first part of the proposition.

Since, by Corollary 3.4, the sectional curvature and diameter are uniformly bounded on  $\mathfrak{FMM}$ , the proof of the second part is identical.

QED

Note that since, for  $X = Y/O(n)$ ,  $\lambda_p^X = \lambda_p^O$ , the estimate (67) is valid for the operator  $\Delta_O$  on  $L_O^2(Y)$ .

Next, we obtain estimates for the corresponding eigenfunctions in various functional spaces. Recall that we use the notation  $\phi_p(x)$  for the eigenfunctions on  $X$ ,  $\widetilde{\phi}_p(y)$  for the eigenfunctions on  $Y$  and  $\phi_p^O$  for the eigenfunctions

of  $\Delta_O$  on  $Y$ . Since our results are independent of the parameter  $p$  but rather the upper bound for  $\lambda_p$ , we will often skip subindex  $p$ .

We start with the moduli space  $\mathfrak{FM}$ .

**Proposition 5.3** *For any  $E > 0$ , there are  $C_F = C_F(n, \Lambda, D) > 0$ ,  $s_F = s_F(n, \Lambda, D) > 0$  such that, for any  $(FM, \mu_{FM}) \in \mathfrak{FM}$ , and  $\tilde{\phi}$  satisfying*

$$-\Delta_Y \tilde{\phi} = \lambda \tilde{\phi}, \quad \lambda \leq E, \quad (70)$$

*we have*

$$\tilde{\phi} \in C_*^3(FM) \quad (71)$$

*and*

$$\|\tilde{\phi}\|_{C^{0,1}(FM)} \leq C_F(1 + E^2)^{s_F/2} \|\tilde{\phi}\|_{L^2(FM, \mu_{FM})}. \quad (72)$$

The proof of the proposition is based on the following result:

**Proposition 5.4** *There is a monotonously increasing function,  $C_1 = C_1(n, \Lambda, D) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that, for any  $(FM, \mu_{FM}) \in \mathfrak{FM}$ ,  $y \in FM$ , and  $r > 0$*

$$\mu_{FM}(B(y, r)) \geq C_1(r). \quad (73)$$

**Proof.** Since for  $(FM, \mu_{FM}) \in \mathfrak{FM}$ ,  $\dim(FM) = \ell = n + \dim(O(n))$ , by the Bishop-Gromov theorem,

$$\frac{V_{\tilde{h}}(B(y, r))}{V_{\tilde{h}}(B(y, D_F))} \geq \frac{C^\ell(r)}{C^\ell(D_F)},$$

where  $\tilde{h}$  is the riemannian metric on  $FM$ . Here  $D_F = \sup(\text{diam}(FM))$ , when  $M \in \mathfrak{MM}$ ,  $C^\ell(r)$  is the volume of a ball of radius  $r$  in the  $\ell$ -dimensional simply connected manifold with constant sectional curvature  $-\Lambda_F$ , where  $\Lambda_F$  is the uniform bound for the sectional curvature on  $\mathfrak{MM}$ , see (43). Since

$$V_{\tilde{h}}(B(y, D_F)) = V_{\tilde{h}}(FM), \quad \frac{V_{\tilde{h}}(B(y, r))}{V_{\tilde{h}}(FM)} = \mu_{\tilde{h}}(B(y, r)),$$



this implies (73) with

$$C_1(r) = \frac{C^\ell(r)}{C^\ell(D_F)}.$$

QED

**Proof** (of Prop. 5.3.) For  $y \in FM$ , let  $\widehat{B}_y(r) \subset T_y(FM)$  be a closed ball of radius  $r < \pi/\Lambda_F$ . Let  $\widehat{h}$  denote the induced metric  $(\exp_x)^*\widetilde{h}$  on  $\widehat{B}_y(r)$ . By [2], there are  $C_0 > 0$  and  $0 < r_0 < \pi/\Lambda_F$ , independent on  $FM \in \mathfrak{FMM}$ , such that there are harmonic coordinates  $\Psi : \widehat{B}_y(r_0) \rightarrow \mathbb{R}^\ell$ . In these coordinates, the metric tensor satisfies

$$C_0^{-2}I \leq \Psi_*\widehat{h} \leq C_0^{-2}I, \quad \|\Psi_*\widehat{h}\|_{C_*^2(\Psi(\widehat{B}_y(R_0)))} \leq C_0. \quad (74)$$

In following, let  $R_0 = r_0/2C_0$ . Then the ball  $B(R_0) \subset \mathbb{R}^\ell$  satisfies  $B(R_0) \subset \Psi(\widehat{B}(r_0/2))$ . Next, we denote  $\widehat{\phi} = \Psi_*(\widetilde{\phi})$ , keeping the notation  $\widehat{h}$  for the metric tensor in these coordinates. Below, we use the Lebesgue spaces  $L^p(B(R))$  and Sobolev spaces  $H^s(B(R))$  which are defined using the usual Lebesgue measure on  $B(R_0) \subset \mathbb{R}^\ell$ . We note that as the metric  $\widehat{h}$  satisfies (74), the norms of these spaces are equivalent to the norms in the spaces defined using  $dV_{\widehat{h}}$ .

As  $\widetilde{\phi}$  satisfies (70), we see that e.g. for  $R_1 = (1 - (8\ell)^{-1})R_0$

$$\Delta_{\widehat{h}}\widehat{\phi} = \lambda\widehat{\phi} \quad \text{in } B(R_0). \quad (75)$$

Let us now consider radii  $R_k = (1 - k/(8\ell))R_0$  and  $q_k$  given by  $1/q_k = 1/2 - k/(2\ell)$ ,  $k = 0, 1, \dots, \ell$ . By Sobolev's embedding theorem,  $W^{2,q_{k-1}}(B(R_{k-1})) \subset L^{q_k}(B(R_{k-1}))$  and thus, the local elliptic regularity estimates, see e.g. [45, Thm. 9.4.1], and estimates (74) imply that  $\widehat{\phi} \in W^{2,q_k}(B(R_k))$  and

$$\|\widehat{\phi}\|_{W^{2,q_k}(B(R_k))} \leq c_{k,\ell}(1 + E^2)^{1/2}\|\widehat{\phi}\|_{W^{2,q_{k-1}}(B(R_{k-1}))}, \quad (76)$$

where the constant  $c_{k,\ell} > 0$ , as well as the other constants appearing later in this section, are uniform for  $FM \in \mathfrak{FMM}$ . Combining the estimates (76) with  $k = 0, 1, \dots, \ell$ , together and using Sobolev embedding theorem once more, we obtain that  $\widehat{\phi} \in C^1(B(R_0/2))$ , and

$$\|\widehat{\phi}\|_{C^1(\overline{B}(R_0/2))} \leq C_1(\ell)(1 + E^2)^{s_1/2}\|\widehat{\phi}\|_{L^2(B(R_0))}, \quad (77)$$

where  $s_1 = \ell + 1$ . These provide estimate (77), with  $s_1 = \ell + 1$  for any  $FM \in \mathfrak{FMM}$ .

Next observe that  $\widehat{\phi}$  satisfies

$$\widehat{h}^{ij} \partial_i \partial_j \widetilde{\phi} = -\frac{1}{\sqrt{|\widehat{h}|}} \partial_i \left( \sqrt{|\widehat{h}|} \widehat{h}^{ij} \right) \partial_j \widehat{\phi} - \lambda \widehat{\phi}, \quad (78)$$

where the right side is uniformly bounded in  $C^0(\overline{B}(R_0/2))$ .

Using the same interpolation arguments as in Lemma 4.2, see also [2], [61], [64], we obtain that  $\widehat{\phi} \in C_*^2(\overline{B}(R_0/4))$  and

$$\|\widehat{\phi}\|_{C_*^2(\overline{B}(R_0/4))} \leq c(1 + E)^{(s_1+1)/2} \|\widehat{\phi}\|_{L^2(\mathbb{B}(R_0))}.$$

Therefore, we see that the right side of (78) is uniformly bounded in  $C_*^1(\overline{B}(R_0/4))$ . Again using the same interpolation type arguments as in Lemma 4.2, or [2], we see that  $\widehat{\phi} \in C_*^3(\overline{B}(R_0/8))$  and

$$\|\widehat{\phi}\|_{C_*^3(\overline{B}(R_0/8))} \leq c(1 + E)^{s_1/2+1} \|\widehat{\phi}\|_{L^2(\mathbb{B}(R_0))}. \quad (79)$$

Here and later by  $c$ ,  $C$ , etc we denote various positive constants uniform on  $\mathfrak{FMM}$ .

Thus, to prove estimate (72) it is sufficient to show that  $\|\widehat{\phi}\|_{L^2(B(R_0))}$  is uniformly bounded if  $\|\widetilde{\phi}\|_{L^2(FM, \mu_{FM})}$  is uniformly bounded.

To this end we introduce  $N(G(y))$ , which is the order of the local pseudo-group  $G = G(y)$  corresponding to  $y \in M$  that acts on  $B(r)$ ,  $0 < r < r_0/2$ . Recall that such pseudo-group was introduced in subsection 2.1 and where it was denoted  $G_i$  being associated with manifold  $M_i \in \mathfrak{MM}$ .

Let now  $\Omega^f \subset \widehat{B}(r)$  be an open fundamental domain of  $\widehat{B}(r)/G$ , namely,

$$\Omega^f = \{\widehat{y} \in \widehat{B}(r); d_{\widehat{h}}(O, \widehat{y}) < d_{\widehat{h}}(g(O), \widehat{y}) \text{ for all } g \in G(y), g \neq \text{id}\}.$$

Then,

$$V_{\widehat{h}}(\Omega^f) = V_{\widehat{h}}(B(r)), \quad \int_{\Omega^f} \widehat{u}(\widehat{y}) dV_{\widehat{h}}(\widehat{y}) = \int_{B(y,r)} u(y) dV_{\widehat{h}}(y),$$

for any  $u \in L^1(B(y, r))$ ,  $\widehat{u} = \exp_x^*(u)$ . We then have

$$\|\widehat{\phi}\|_{L^2(\widehat{B}(r), dV_{\widehat{h}})}^2 \leq N(G(y)) \int_{\Omega^f} |\widehat{\phi}|^2 dV_{\widehat{h}} = N(G(y)) \int_{B(y,r)} |\widetilde{\phi}|^2 dV_{\widehat{h}}.$$

Since

$$g_1(\Omega^f) \cap g_2(\Omega^f) = \emptyset, \quad \text{if } g_1 \neq g_2, \quad \text{and } g(\Omega^f) \subset \widehat{B}(2r), \quad \text{for } g \in G,$$

we have

$$V_{\widehat{h}}(\widehat{B}(2r)) \geq N(G(y))V_{\widehat{h}}(B(x, r)).$$

Therefore, for  $0 < r < r_0/2$ ,

$$\begin{aligned} \|\widehat{\phi}\|_{L^2(\widehat{B}(r), dV_{\widehat{h}})}^2 &\leq \frac{V_{\widehat{h}}(\widehat{B}(2r))}{V_{\widehat{h}}(B(y, r))} \int_{B(y, r)} |\widetilde{\phi}|^2 dV_{\widehat{h}} \\ &\leq V_{\widehat{h}}(\widehat{B}(2r)) \frac{V_{\widehat{h}}(FM)}{V_{\widehat{h}}(B(y, r))} \int_{FM} |\widetilde{\phi}|^2 d\mu_{\widehat{h}} \\ &\leq \frac{C(2r)}{C_1(r)} \int_M |\widetilde{\phi}|^2 d\mu_h. \end{aligned}$$

Here, at the last stage, we use (73) and (74). This estimate, together with (79), implies (71). To obtain estimate (72) we observe that

$$\|u\|_{C^{0,1}(FM)} \leq \max_{y \in FM} \|\exp_y^* u\|_{C^{0,1}(B(r_0/8))}.$$

QED

**Remark 5.5** *Using a proper definition of the global  $C^{1,1}$ -spaces, a slightly more delicate analysis shows that estimate (72) is valid for the  $C^{1,1}$ -norm of  $\widetilde{\phi}$ . Moreover, using local definition of  $C_*^3$ -spaces, we can obtain from the above proof that the uniform estimate (72) remains valid in the local  $C_*^3$ -spaces. However, since our principal goal are the estimates on  $\overline{\mathfrak{MM}}$ , where spaces  $C^{k,\alpha}(X)$ ,  $0 < \alpha \leq 1$ , are not defined for  $k > 0$ , we restrict our considerations to the space  $C^{0,1}$ .*

Here we provide the proof of estimate (72) for  $Y \in \overline{\mathfrak{MM}}$ .

**Lemma 5.6** *There are  $C_F = C(n, \Lambda, D)$ ,  $s_F = s(n, \Lambda, D)$  such that, for any  $E > 0$ ,  $(Y, \mu_Y) \in \overline{\mathfrak{MM}}$ ,*

$$\|\widetilde{\phi}\|_{C^{0,1}(Y)} \leq C_F(1 + E^2)^{s_F/2} \|\widetilde{\phi}\|_{L^2(Y, \mu_Y)}, \quad (80)$$

cf. (72)

**Proof.** Due to (42) and Proposition 5.3, it is enough to consider  $Y \in \overline{\mathfrak{FMM}} \setminus \mathfrak{FMM}$ . Let  $(Y, \mu_Y)$  be the limit of  $(FM_k, \tilde{\mu}_k) \in \mathfrak{FMM}$  in the measured GH topology and let

$$\varepsilon_k = d_{pmGH}(Y, FM_k).$$

Then,  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . Denote by  $\tilde{f}_k : FM_k \rightarrow Y$  a regular fibration of  $Y$  which enjoys properties (64) and we take into account that the existence of such fibration for  $Y$  is analogous to that for  $X$ .

Let  $\tilde{\phi}$  be an eigenfunction of  $\Delta_Y$ , corresponding to an eigenvalue  $\lambda < E$ , i.e.,

$$-\Delta_Y \tilde{\phi} = \lambda \tilde{\phi}, \quad \|\tilde{\phi}\|_{L^2(Y)} = 1.$$

By [24], there exists  $\delta_k > 0$  such that  $\lim_{k \rightarrow \infty} \delta_k = 0$  and there are

$$\tilde{\phi}_k \in \mathcal{L}_k(\lambda - \delta_k, \lambda + \delta_k), \quad \|\tilde{\phi}_k\|_{L^2(FM_k)} = 1,$$

satisfying

$$\|\tilde{f}_k^*(\tilde{\phi}) - \tilde{\phi}_k\|_{L^2(FM_k)} < \delta_k. \quad (81)$$

Here and later, for  $a < b$ ,  $a, b \notin \text{spec}(-\Delta_{FM_k})$ , we denote by  $\tilde{\mathcal{L}}_k(a, b) = \tilde{\mathcal{L}}_{FM_k}(a, b)$  linear combination of the eigenfunctions of  $-\Delta_{FM_k}$  corresponding to the eigenvalues in the interval  $(a, b)$  and define similar spaces  $\tilde{\mathcal{L}}_Y(a, b)$  for  $Y \in \overline{\mathfrak{FMM}}_p$ .

Let  $\tilde{f}'_k$  be a (non-necessary continuous) right inverse to  $\tilde{f}_k$ ,

$$\tilde{f}_k \circ \tilde{f}'_k = \text{id}_Y.$$

Denote

$$\tilde{\phi}'_k(y) = \tilde{\phi}_k(\tilde{f}'_k(y)), \quad y \in Y.$$

Observe that

$$\|\tilde{f}_k^*(\tilde{\phi}'_k) - \tilde{\phi}_k\|_{L^\infty(FM_k)} \leq c(1 + E^2)^{s/2} \varepsilon_k, \quad (82)$$

where  $c = c(E)$ ,  $s = s(E)$  are uniform on  $\mathfrak{FMM}$ . Indeed, for any  $y_k, y'_k \in \tilde{f}_k^{-1}(y)$  formula (64) yields  $d_{FM_k}(y_k, y'_k) \leq c\varepsilon_k$ . This, together with the uniform Lipschitz continuity of functions  $\phi_{FM_k}$ , see (67) and (72), yield

$$\|\tilde{\phi}_k\|_{C^{0,1}(FM_k)} \leq c(1 + E^2)^{s/2} \|\tilde{\phi}_k\|_{L^2(FM_k)}, \quad \tilde{\phi}_k \in \tilde{\mathcal{L}}_{FM_k}(-1, E). \quad (83)$$

This implies the inequality (82). Therefore,

$$\|\tilde{f}_k^*(\tilde{\phi}'_k) - \tilde{\phi}_k\|_{L^2(FM_k)} \leq c(1 + E^2)^{s/2} \varepsilon_k,$$

where  $c$  and  $s$  are uniform constants on  $\overline{\mathfrak{FMM}}_p$ . In view of (81), this implies that

$$\|\tilde{f}_k^*(\tilde{\phi}'_k - \tilde{\phi})\|_{L^2(FM_k)} \leq c(1 + E^2)^{s/2} (\delta_k + \varepsilon_k). \quad (84)$$

On the other hand, there exists a subsequence,  $k = k(p)$ , and a function  $\phi' \in C^{0,1}(Y)$  such that, for any  $y \in Y$ ,

$$\lim_{k \rightarrow \infty} \tilde{\phi}'_k(y) = \tilde{\phi}'(y).$$

Indeed, choosing a dense subset  $\{y_p\}_{p=1}^\infty \subset Y$  and using the diagonalization procedure we find  $k = k(n)$ ,  $k(n) \rightarrow \infty$  as  $n \rightarrow \infty$  such that for all  $p \in \mathbb{Z}_+$  there exists limits

$$\tilde{\phi}'(y_p) = \lim_{n \rightarrow \infty} \tilde{\phi}'_{k(n)}(y_p).$$

Using the uniform Lipschitz bound (83), the estimate  $d_{FM_k}(y_k, y'_k) \leq c\varepsilon_k$ , for  $y_k, y'_k \in \tilde{f}_k^{-1}(y)$ , and the definition of the GH distance, i.e. the first equations in (64), we see that

$$|\tilde{\phi}'_{k(n)}(y) - \tilde{\phi}'_{k(n)}(y')| \leq c(1 + E^2)^{s/2} (\varepsilon_{k(n)} + d_Y(y, y')), \quad y, y' \in Y.$$

Thus, we extend  $\tilde{\phi}'$  from  $\{y_p : p \in \mathbb{Z}_+\} \subset Y$  to the whole space  $Y$  so that

$$|\tilde{\phi}'(y) - \tilde{\phi}'(y')| \leq c(1 + E^2)^{s/2} d_Y(y, y'), \quad \text{for all } y, y' \in Y. \quad (85)$$

Moreover, the above two inequalities yield that

$$\|\tilde{\phi}'_{k(n)} - \tilde{\phi}'\|_{L^\infty(Y)} \leq c(1 + E^2)^{s/2} \varepsilon_{k(n)},$$

so that

$$\|\tilde{f}_{k(n)}^*(\tilde{\phi}'_{k(n)} - \tilde{\phi}')\|_{L^\infty(FM_{k(n)})} \leq c(1 + E^2)^{s/2} \varepsilon_{k(n)}.$$

Comparing this inequality with (84), we see that

$$\|\tilde{f}_{k(n)}^*(\tilde{\phi} - \tilde{\phi}')\|_{L^2(FM_{k(n)})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (86)$$

Recall that  $\tilde{\phi}$  satisfies (70) where the coefficients of  $\Delta_Y$  satisfy  $\tilde{h}, \rho_Y \in C_*^2(Y)$  and thus  $\phi \in C_*^3(Y)$ . This, (86) and the second equation in (64) corresponding to the measured convergence  $(FM_k, \tilde{\mu}_k) \rightarrow (Y, \mu_Y)$  implies that  $\tilde{\phi}(y) = \tilde{\phi}'(y)$  for all  $y \in Y$ . By (85), this yields (80).

QED

Considering  $\overline{\mathfrak{M}\mathfrak{M}}$ , we obtain

**Corollary 5.7** *For any  $E > 0$ ,  $(X, p, \mu_X) \in \overline{\mathfrak{M}\mathfrak{M}}_p$ , and eigenfunction  $\phi$ ,*

$$-\Delta_X \phi = \lambda \phi, \quad \lambda \leq E,$$

*we have*

$$\|\phi\|_{C^{0,1}(X)} \leq C_F(1 + E^2)^{s_F/2} \|\phi\|_{L^2(X, \mu_X)}, \quad (87)$$

*where constants  $C_F, s_F$  are the same as in Proposition 5.3.*

**Proof** Estimate (87) follows from (72) if we take into account (48), (240) and (243).

QED

Analysing the proof of Proposition 5.3, Lemma 5.6 and Corollary 5.7, we obtain

**Corollary 5.8** *Let  $\tilde{u} \in \tilde{\mathcal{L}}_Y(-\infty, a)$ ,  $a \notin \text{spec}(-\Delta_Y)$  or  $u \in \mathcal{L}_X(-\infty, a)$ ,  $a \notin \text{spec}(-\Delta_X)$ . Then estimates (80) and (87) remain valid for  $\tilde{u}$ ,  $u$ , correspondingly.*

## 5.2 Spectral convergence on $\overline{\mathfrak{F}\mathfrak{M}\mathfrak{M}}$ and $\overline{\mathfrak{M}\mathfrak{M}}$

Recall that if  $M_k \rightarrow X$  in  $\overline{\mathfrak{M}\mathfrak{M}}_p$ , i.e. in the pointed measured GH-topology, then  $FM_k \rightarrow Y$  in  $\overline{\mathfrak{F}\mathfrak{M}\mathfrak{M}}$ , with  $X = Y/O(n)$ . Moreover, we can choose  $C^{0,1}$  regular approximations,  $\tilde{f}_k, f_k$  which satisfy properties described in Theorem 2.5.

Using the pull back  $f_k^*$  of  $f_k$  and  $\tilde{f}_k^*$  of  $\tilde{f}_k$ , we define

$$\begin{aligned} \mathcal{L}_k^*(a, b) &= f_k^*(\mathcal{L}_X(a, b)) \subset C^{0,1}(M_k) \subset L^2(M_k), \\ \tilde{\mathcal{L}}_k^*(a, b) &= \tilde{f}_k^*(\tilde{\mathcal{L}}_Y(a, b)) \subset C^{0,1}(FM_k) \subset L^2(FM_k) \end{aligned}$$

For linear space  $\mathcal{Z} \subset L^2(X)$  we denote by  $\mathcal{B}(\mathcal{Z})$  the unit ball of  $\mathcal{Z}$ , that is, the intersection of  $\mathcal{Z}$  and the unit ball of  $L^2(X)$  and similarly for  $\mathcal{Z} \subset L^2(Y)$ .

**Proposition 5.9** 1. Let  $\lim_{k \rightarrow \infty} (FM_k, \tilde{\mu}_k) = (Y, \mu_Y)$  in the measured GH topology. Assume that  $a, b \in \mathbb{R}$  satisfy  $a < b$  and  $a, b \notin \text{spec}(-\Delta_Y)$ . Then, for any  $0 \leq \alpha < 1$ ,

$$\tilde{\delta}_k^\alpha := d_{C^\alpha(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_k(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (88)$$

2. Let  $\lim_{k \rightarrow \infty} (M_k, \mu_k) = (X, \mu_X)$  in the measured GH topology. Assume that  $a, b \in \mathbb{R}$  satisfy  $a < b$  and  $a, b \notin \text{spec}(-\Delta_X)$ . Then, for any  $0 \leq \alpha < 1$ , we have

$$\delta_k^\alpha := d_{C^\alpha(M_k)} \left( \mathcal{B}(\mathcal{L}_k^*(a, b)), \mathcal{B}(\mathcal{L}_k(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (89)$$

Here and later, if  $S_1, S_2 \subset \mathcal{M}$ , where  $\mathcal{M}$  is a metric space, then  $d_{\mathcal{M}}(S_1, S_2)$  stands for the Hausdorff distance between  $S_1$  and  $S_2$ .

**Proof.** 1. It follows from [24] that, for large  $k$ , we have  $a, b \notin \text{spec}(\Delta_{FM_k})$ ,  $\dim(\tilde{\mathcal{L}}_k(a, b)) = \dim(\tilde{\mathcal{L}}_k^*(a, b))$ , and

$$\varepsilon_k := d_{L^2(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_k(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (90)$$

Our aim is to show that this implies that  $\tilde{\delta}_k^\alpha \rightarrow 0$ .

We start with  $\alpha = 0$  and assume that (88) does not hold. Thus, without loss of generality, there is  $\tilde{\delta}_k = \tilde{\delta}_k^0$  such that, for any  $k$  there either exists  $\tilde{u}_k \in \mathcal{B}(\tilde{\mathcal{L}}_k(a, b))$  such that

$$\|\tilde{u}_k - \tilde{v}^*\|_{C(FM_k)} > \frac{\tilde{\delta}_k}{2}, \quad \text{for all } \tilde{v}^* \in \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \quad (91)$$

or there exists  $\tilde{v}_k^* \in \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b))$  such that

$$\|\tilde{v}_k^* - \tilde{u}\|_{C(FM_k)} > \frac{\tilde{\delta}_k}{2}, \quad \text{for all } \tilde{u} \in \mathcal{B}(\tilde{\mathcal{L}}_k(a, b)).$$

Without loss of generality we can consider the case where there exists  $\tilde{u}_k \in \mathcal{B}(\tilde{\mathcal{L}}_k(a, b))$  such that (91) hold for all  $k$ . On the other hand, by (90), for all  $k$  there are  $\tilde{v}_k^* \in \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b))$  such that

$$\|\tilde{u}_k - \tilde{v}_k^*\|_{L^2(FM_k)} < 2\varepsilon_k. \quad (92)$$

However, by (91), there is  $y_k \in FM_k$  with  $|\tilde{u}_k(y_k) - \tilde{v}_k^*(y_k)| > \delta_k/2$ . Due to the GH convergence of  $FM_k$  to  $Y$ , uniform Lipschitz estimate (72), (80), together with (64), imply that there is  $c_0 = C(1 + b^2)^{s/2}$  such that

$$|\tilde{u}_k(y) - \tilde{v}_k^*(y)| > \frac{\tilde{\delta}_k^0}{4}, \quad \text{for all } y \in FM_k \text{ with } \tilde{d}_k(y_k, y) < \frac{1}{4}c_0\tilde{\delta}_k.$$

Therefore, taking into account that  $\rho_k = 1$  on  $FM_k$  and using (73), there is a uniform constant  $c_1 > 0$ , such that

$$\|\tilde{u}_k - \tilde{v}_k^*\|_{L^2(FM_k, \tilde{\mu}_k)}^2 > c_1 C_1 \left( \frac{1}{4}c_0\tilde{\delta}_k^0 \right) (\tilde{\delta}_k)^2.$$

Comparing this estimate with (92) we see that, with some  $C(b)$ ,

$$\tilde{\delta}_k \leq C(b)\varepsilon_k^{2/(n+2)}, \quad (93)$$

proving (88).

On the other hand, it follows from Theorem 2.5 (2), that, for large  $k$

$$\|\mathcal{B}(\tilde{\mathcal{L}}_k^*(-\infty, E))\|_{C^{0,1}(FM_k)} \leq 2C_F(1 + E^2)^{s_F/2}.$$

Interpolating this inequality with (93), we see that  $\tilde{\delta}_k^\alpha \rightarrow 0$ , as  $k \rightarrow \infty$ ,  $\alpha < 1$ .

2. Due to the fact that  $\tilde{f}_k$  are  $O(n)$ -maps, see Theorem 2.5,

$$\tilde{f}_k^* : L_O^2(Y) \rightarrow L_O^2(FM_k).$$

Therefore, it follows from [24] that

$$d_{L_O^2(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_{k,O}^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_{k,O}(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

if  $a, b \notin \text{spec}(-\Delta_Y)$ . Here e.g.  $\tilde{\mathcal{L}}_{k,O}(a, b)$  is the subspace of  $L_O^2(FM_k)$  spanned by the  $O(n)$ -invariant eigenfunctions of  $-\Delta_{FM_k}$ . Using the same arguments as above, we see that, for any  $0 \leq \alpha < 1$ , estimate (88) remains valid for  $\tilde{\mathcal{L}}_k^*(a, b)$  and  $\tilde{\mathcal{L}}_k(a, b)$  replaced by  $\tilde{\mathcal{L}}_{k,O}^*(a, b)$  and  $\tilde{\mathcal{L}}_{k,O}(a, b)$ .

In view of Theorem 2.5 these imply estimate (89) due to (238) and (243).

QED

Consider now the general case and assume that

$$d_{pmGH}((X_k, p_k, \mu_k), (X, p, \mu_X)) \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad (94)$$



where  $X_k, X \in \overline{\mathfrak{MM}}_p$ . Our goal is to obtain an estimate similar to (89) for this case. As for the case of  $(X, p, \mu_X) = \lim_{pmGH} (M_k, p_k, \mu_k)$ , we start with analysing the case

$$(Y, \tilde{\mu}_Y) = \lim_{k \rightarrow \infty} (Y_k, \mu_Y^k) \quad \text{in the mGH-topology,}$$

where we write  $\tilde{\mu}_Y^k$  for the measure on  $Y_k$ .

**Lemma 5.10** *1. Assume  $Y \in \overline{\mathfrak{MM}}_p$ ,  $a, b \notin \text{spec}(-\Delta_Y)$ , and  $Y_k \in \overline{\mathfrak{MM}}_p$  converge with measure to  $Y$ . Then, for any  $0 \leq \alpha < 1$ ,*

$$\tilde{\delta}_k^\alpha := d_{C^\alpha(Y_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_k(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (95)$$

*2. Assume  $X \in \overline{\mathfrak{MM}}_p$ ,  $a, b \notin \text{spec}(-\Delta_X)$  and  $X_k \in \overline{\mathfrak{MM}}_p$  converge to  $X$  in the sense of (94). Then, for any  $0 \leq \alpha < 1$ ,*

$$\delta_k^\alpha := d_{C^\alpha(X_k)} \left( \mathcal{B}(\mathcal{L}_k^*(a, b)), \mathcal{B}(\mathcal{L}_k(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (96)$$

**Proof.** *1.* As earlier, we start with the case  $\alpha = 0$ ,  $\tilde{\delta}_k = \tilde{\delta}_k^0$ . Assuming the opposite, we can assume, without loss of generality, that there is  $\tilde{\delta}_0 > 0$  such that

$$d_{C(Y_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_k(a, b)) \right) > \tilde{\delta}_0, \quad \text{for all } k \geq 1. \quad (97)$$

Approximate, in the measured GH-topology,  $(Y_k, \tilde{\mu}_Y^k)$  by  $(FM_k, \tilde{\mu}_k) \in \overline{\mathfrak{MM}}_p$ , and let  $\Delta_{kk}$  stands for the Laplacian on  $FM_k$ . Denote by  $\tilde{\mathcal{L}}_{kk}(a, b) \subset L^2(FM_k)$  the eigenspace of  $-\Delta_{kk}$  corresponding to the eigenvalues in  $(a, b)$ , and by

$$\tilde{\mathcal{L}}_{kk}^*(a, b) = f_{kk}^* \left( \tilde{\mathcal{L}}_k(a, b) \right),$$

where  $f_{kk} : FM_k \rightarrow Y_k$  is the regular approximation described in Theorem 2.5, (28) with  $\varepsilon_k$  changed into  $\varepsilon_{kk}$ . Moreover, we can assume that

$$d_{L^2(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_{kk}^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_{kk}(a, b)) \right) < \frac{1}{k}. \quad (98)$$

Let

$$\tilde{f}_k : Y_k \rightarrow Y,$$

be a  $\tilde{\varepsilon}_k$ -Riemannian submersion, satisfying (28) and providing the measured GH-convergence of  $Y_k$  to  $Y$ , which is guaranteed by Corollary 2.7, (1), with  $Y_k, Y$  instead of  $X_k, X$ . We can assume  $\tilde{f}_k$  to be an  $O(n)$ -map.

Consider

$$\tilde{f}_{kk} = \tilde{f}_k \circ f_{kk} : FM_k \rightarrow Y,$$

which provides a regular  $\tilde{\varepsilon}_{kk}$ -Riemannian submersion of  $FM_k$  to  $Y$ ,  $\tilde{\varepsilon}_{kk} \rightarrow 0$ , as  $k \rightarrow \infty$ . Therefore,

$$\lim_{k \rightarrow \infty} d_{L^2(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_{kk}(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_{kk}^*(a, b)) \right) = 0,$$

where  $\tilde{\mathcal{L}}_{kk}(a, b)$  is the eigenspace of  $-\Delta_{kk}$  corresponding to the interval  $(a, b)$  and

$$\tilde{\mathcal{L}}_{kk}^*(a, b) = \tilde{f}_{kk}^*(\tilde{\mathcal{L}}_k(a, b)) = f_{kk}^*(\tilde{\mathcal{L}}_k^*(a, b)). \quad (99)$$

This, together with (98) implies that

$$\lim_{k \rightarrow \infty} d_{L^2(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_{kk}^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_{kkk}^*(a, b)) \right) = 0.$$

On the other hand, estimate (97) together with definition (99) yields that, for large  $k$ ,

$$d_{C(FM_k)} \left( \mathcal{B}(\tilde{\mathcal{L}}_{kk}^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}_{kkk}^*(a, b)) \right) > \tilde{\delta}_0/2.$$

Similar arguments to used in the proof of part (1) of Proposition 5.9 show that the above two equalities lead to a contradiction. This proves (95) for  $\alpha = 0$ .

To obtain the result for any  $0 \leq \alpha < 1$ , we use again the fact that, due to Corollaries 5.8 and 2.7, (1), there is  $c_F^b > 0$  such that, for large  $k$ ,

$$\|\mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b))\|_{C^{0,1}(Y_k)} \leq c_F^b, \quad \|\mathcal{B}(\tilde{\mathcal{L}}_k(a, b))\|_{C^{0,1}(Y_k)} \leq c_F^b.$$

(2). Since  $\tilde{f}_k$  is an  $O(n)$ -map, estimate (95) remains valid with  $\tilde{\mathcal{L}}_{k,O}^*(a, b)$  and  $\tilde{\mathcal{L}}_{k,O}(a, b)$  instead of  $\tilde{\mathcal{L}}_k^*(a, b)$ ,  $\tilde{\mathcal{L}}_k(a, b)$ .

Denote  $\pi_k : Y_k \rightarrow X_k = Y_k/O(n)$ . Then (95) for the  $O(n)$ -invariant functions, together with (240) and (241) proves (96).

QED

**Remark 5.11** If  $X = \lim_{pmGH} X_k = \{point, 1\}$ , then  $L^2(X) = \mathbb{R}$  and  $\Delta_X = 0$ . Thus, the only eigenvalue is  $\lambda_0 = 0$  with the corresponding eigenfunction 1. Due to (68), (96) remains trivially valid with, when  $k$  is sufficiently large, with  $\tilde{\mathcal{L}}_k(a, b)$ ,  $\tilde{\mathcal{L}}_k^*(a, b)$  consisting of constant functions, if  $0 \in (a, b)$ , and of only 0-function, if  $0 \notin (a, b)$ .

Denote by  $f'_k : X \rightarrow X_k$  the almost right inverse to  $f_k$ , i.e.  $f_k, f'_k$  satisfy conditions (64) and

$$d_k(x_k, f'_k \circ f_k(x_k)), d_X(x, f_k \circ f'_k(x)) = \varepsilon_k \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (100)$$

Then using (96) and the uniform  $C^{0,1}$ -boundedness of  $\mathcal{B}(\mathcal{L}_k(a, b))$ ,  $\mathcal{B}(\mathcal{L}(a, b))$ , see Corollary 5.7, we obtain

**Corollary 5.12** *Let  $(X_k, p_k, \mu_k), (X, p, \mu_X) \in \overline{\mathfrak{MM}}_p$ , be such that  $(X, p, \mu_X) = \lim_{k \rightarrow \infty} (X_k, p_k, \mu_k)$  in the pmGH-topology. Then, there exists  $\sigma_k \rightarrow 0$ , as  $k \rightarrow \infty$ , such that*

$$d_{L^\infty(X)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}(a, b)) \right) < \sigma_k, \quad (101)$$

where

$$\tilde{\mathcal{L}}_k^*(a, b) = (f'_k)^*(\mathcal{L}_k(a, b)).$$

Moreover, there is  $c > 0$ , such that if  $d_X(x, x') < \sigma_k$  then for all  $u_k^* \in \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b))$  we have

$$|u_k^*(x) - u_k^*(x')| < c\sigma_k. \quad (102)$$

**Proof** The first claim follows from by uniform  $C^{0,1}$ -continuity of  $\mathcal{B}(\tilde{\mathcal{L}}(a, b))$  together with (96) for  $\alpha = 0$  and (100).

To prove (102) we also use the uniform  $C^{0,1}$ -continuity in  $\mathcal{L}_k(a, b)$ ,  $\mathcal{L}(a, b)$  together with (101).

**Remark 5.13** Added in proof to [24] is the note that the given formulation of Theorem (0.4) (C) is incorrect. However, it follows from Corollaries 5.8 and 5.12 that, for our special choice of  $f'_k$ ,

$$d_{L^2(X)} \left( \mathcal{B}(\tilde{\mathcal{L}}_k^*(a, b)), \mathcal{B}(\tilde{\mathcal{L}}(a, b)) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Our next goal is to obtain continuity of the eigenfunctions in  $L^\infty$ -norm, with respect to the measured GH distance which is uniform on  $\overline{\mathfrak{MM}}_p$ .

**Definition 5.14** *Let  $(a_\ell, b_\ell)$ ,  $\ell = 1, \dots, L$ , be a finite collection of open intervals, and  $d_\ell, \ell = 1, \dots, L$ , be positive numbers. Denote the set  $(a_\ell, b_\ell, d_\ell)_{\ell=1}^L$  by  $\mathcal{I}$ . Then we define*

$$\overline{\mathfrak{MM}}_{\mathcal{I}} = \{X \in \overline{\mathfrak{MM}}_p : d(\text{spec}(X), \{a_\ell\}) \geq d_\ell \text{ and } d(\text{spec}(X), \{b_\ell\}) \geq d_\ell\}.$$

Note that  $\overline{\mathfrak{MM}}_{\mathcal{I}}$  is closed and thus compact with respect to the pointed measured GH-topology.

**Corollary 5.15** *Let  $\mathcal{I} = (a_\ell, b_\ell, d_\ell)_{\ell=1}^L$  and  $\mathcal{J} = (a_\ell, b_\ell, d'_\ell)_{\ell=1}^L$ ,  $d_\ell > d'_\ell$ . Then, for any  $\varepsilon > 0$  there is  $\sigma = \sigma_{\mathcal{I}, \mathcal{J}}(\varepsilon) > 0$  such that, if  $X \in \overline{\mathfrak{MM}}_{\mathcal{I}}$  and  $d_{pmGH}(X, X') < \sigma$ , then  $X' \in \overline{\mathfrak{MM}}_{\mathcal{J}}$  and  $X$  and  $X'$  satisfy*

$$\dim(\mathcal{L}_X(a_\ell, b_\ell)) = \dim(\mathcal{L}_{X'}(a_\ell, b_\ell)) := n(\ell).$$

Moreover, there are measurable maps  $f : X \rightarrow X'$ ,  $f' : X' \rightarrow X$ , which are  $\varepsilon$ -approximations satisfying (64) and (100), with  $\varepsilon$  in place of  $\varepsilon_k$ , such that

$$\begin{aligned} d_{L^\infty(X)}(\mathcal{B}(f^*(\mathcal{L}_{X'}(a_\ell, b_\ell))), \mathcal{L}_X(a_\ell, b_\ell)) &< \varepsilon, \\ d_{L^\infty(X')}(\mathcal{B}((f')^*(\mathcal{L}_X(a_\ell, b_\ell))), \mathcal{L}_{X'}(a_\ell, b_\ell)) &< \varepsilon. \end{aligned} \quad (103)$$

In addition, for any  $\ell = 1, \dots, L$ , if  $\phi_{i,\ell}, i = 1, \dots, n(\ell)$ , and  $\phi'_{i,\ell}, i = 1, \dots, n(\ell)$ , form an orthonormal eigenfunction basis in  $\mathcal{L}_X(a_\ell, b_\ell)$ ,  $\mathcal{L}_{X'}(a_\ell, b_\ell)$ , correspondingly, then there exist orthogonal matrices

$$\mathbb{U}_\ell = [u_{ij}]_{i,j=1}^{n(\ell)} \in O(n(\ell)), \quad \mathbb{U}'_\ell = [u'_{ij}]_{i,j=1}^{n(\ell)} \in O(n(\ell)),$$

such that  $\phi_{i,\ell}^* = \sum_{j=1}^{n(\ell)} u_{ij} f^*(\phi'_{j,\ell})$  satisfy

$$\|\phi_{i,\ell} - \phi_{i,\ell}^*\|_{L^\infty(X)} < \varepsilon, \quad i = 1, \dots, n(\ell), \quad \ell = 1, \dots, L. \quad (104)$$

Similar result is valid for  $(f')^*(\phi_{j,\ell})$ , if we use  $\mathbb{U}'_\ell$ .

**Proof** By compactness arguments, the first statement of the corollary follows immediately from Theorem (0.4) (A), [24].

To prove the second statement, in particular, (103), assume that there are  $\varepsilon > 0$ ,  $X_k$ ,  $\widehat{X}_k$  and  $\delta_k \rightarrow 0$  such that  $d_{pmGH}(X_k, \widehat{X}_k) < \delta_k$ , however, (103) is not valid. Without loss of generality, we can assume that  $X_k, \widehat{X}_k \rightarrow X$ , with respect to the pointed measured GH-convergence and  $f_k : X_k \rightarrow \widehat{X}_k$ ,  $f'_k : \widehat{X}_k \rightarrow X_k$  which provide  $\delta_k$ -approximation, are of the form

$$f_k = \widehat{f}_{kk} \circ f'_{kk} \quad f'_k = f_{kk} \circ \widehat{f}'_{kk},$$

where

$$f_{kk} : X \rightarrow X_k, \quad f'_{kk} : X_k \rightarrow X \quad \text{and} \quad \widehat{f}_{kk} : X \rightarrow \widehat{X}_k, \quad \widehat{f}'_{kk} : \widehat{X}_k \rightarrow X,$$

provide  $(\delta_k/4)$ -approximation, in the sense of definition (64), of  $X, X_k$  and  $X, \widehat{X}_k$ , correspondingly. Note that, if  $f_{kk}, f'_{kk}$  and  $\widehat{f}_{kk}, \widehat{f}'_{kk}$  are  $\delta/4$ -almost inverse of each other in the sense of (100), then  $f_k, f'_k$  are  $\delta_k$ -almost inverse. Indeed, say,

$$\begin{aligned} d_k(f'_k \circ f_k(x_k), x_k) &= d_k(f_{kk} \circ \widehat{f}'_{kk} \circ \widehat{f}_{kk} \circ f'_{kk}(x_k), x_k) \\ &< d_X(f'_{kk} \circ f_{kk} \circ \widehat{f}'_{kk} \circ \widehat{f}_{kk} \circ f'_{kk}(x_k), f'_{kk}(x_k)) + \frac{1}{4}\delta_k \\ &< d_X(\widehat{f}'_{kk} \circ \widehat{f}_{kk} \circ f'_{kk}(x_k), f'_{kk}(x_k)) + \frac{1}{2}\delta_k \\ &< d_k(f'_{kk}(x_k), f'_{kk}(x_k)) + \frac{3}{4}\delta_k = \frac{3}{4}\delta_k. \end{aligned}$$

As for the measure closedness, by Lemma 2.10,  $(\delta_k/4)$ -closedness of measures  $\mu_X$  and  $\mu_k$  with respect to  $f_{kk}$  and  $\mu_X$  and  $\widehat{\mu}_k$  with respect to  $\widehat{f}_{kk}$  imply  $\delta_k$ -closedness of measures  $\mu_k$  and  $\widehat{\mu}_k$  with respect to  $f_k$ .

Then, by (101), we have

$$d_{L^\infty}(X) \left( \mathcal{B}(f_{kk}^*(\mathcal{L}_k(a_\ell, b_\ell))), \mathcal{B}(\widehat{f}_{kk}^*(\widehat{\mathcal{L}}_k(a_\ell, b_\ell))) \right) \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

where  $\mathcal{L}_k(a_\ell, b_\ell), \widehat{\mathcal{L}}_k(a_\ell, b_\ell)$  stand for  $\mathcal{L}_{X_k}(a_\ell, b_\ell), \widehat{\mathcal{L}}_{\widehat{X}_k}(a_\ell, b_\ell)$ . Using arguments similar to those leading to (102), we show that

$$d_{L^\infty(X_k)} \left( \mathcal{B}(f_k^*(\widehat{\mathcal{L}}_k(a_\ell, b_\ell))), \mathcal{B}(\mathcal{L}_k(a_\ell, b_\ell)) \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

To prove the last statement of the corollary, recall that, due to the measure convergence in (64) with  $\sigma$  on place of  $\varepsilon$  and uniform continuity (87),

$f^*(\phi'_{j,\ell})$ ,  $j = 1, \dots, n(\ell)$ , satisfy

$$\int_X f^*(\phi'_{j,\ell}) f^*(\phi'_{i,\ell}) d\mu_X \rightarrow \delta_{ij}, \quad \text{as } d_{pmGH}(X, X') \rightarrow 0.$$

Together with (103) this implies the existence of  $\mathbb{U}_\ell = \mathbb{U}_\ell(X, X')$ ,  $\ell = 1, \dots, L$ , such that for  $i = 1, \dots, n(\ell)$ ,

$$\|\phi_{i,\ell} - \phi_{i,\ell}^*\|_{L^\infty(X)} \rightarrow 0, \quad \text{as } d_{pmGH}(X, X') \rightarrow 0,$$

which implies (104).

QED

### 5.3 Heat kernel convergence

In this subsection, we consider the continuity of the heat kernel,  $H_X(x, y, t)$  with respect to the pointed measured GH-convergence on  $\overline{\mathfrak{MM}}_p(n, \Lambda, D)$ . Recall that the heat kernel on  $X$  can be written in terms of eigenfunctions and eigenvalues as

$$H_X(x, y, t) = \sum_{p=0}^{\infty} e^{-\lambda_p t} \phi_p(x) \phi_p(y), \quad (105)$$

where the eigenfunctions  $\phi_p$ ,  $p = 0, 1, \dots$ , form an orthonormal basis in  $L^2(X, \mu_X)$  and  $\lambda_0 = 0$  and  $\phi_0 = 1$ .

**Proposition 5.16** *For any  $0 < \sigma < 1$ , there exists  $C(\sigma) > 0$  and  $s_F > 0$ , introduced in (80), such that, for any  $E > 0$  and  $X \in \overline{\mathfrak{MM}}_p$ ,*

$$|H_X(x, \tilde{x}, t) - \sum_{\lambda_p < E} e^{-\lambda_p t} \phi_p(x) \phi_p(\tilde{x})| \leq C(\sigma) [1 + t^{-(n/2+2s_F)}] e^{-(1-\sigma)Et}. \quad (106)$$

**Proof.** For any  $x, \tilde{x} \in X$ ,

$$\left| \sum_{\lambda_p \geq E} \exp(-\lambda_p t) \phi_p(x) \phi_p(\tilde{x}) \right| \leq \sum_{\lambda_p \geq E} \exp(-\lambda_p t) |\phi_p(x)| |\phi_p(\tilde{x})|,$$

where we choose  $(1 - \sigma/2) < \alpha \leq 1$  such that  $\alpha E \notin \text{spec}(-\Delta_X)$ . Using (80), this yields that

$$\left| \sum_{\lambda_p \geq E} \exp(-\lambda_p t) \phi_p(x) \phi_p(\tilde{x}) \right| \leq C_F^2 \int_{\lambda \geq \alpha E} (1 + \lambda^{2s_F}) \exp(-\lambda t) dN_X(\lambda),$$

where constants  $C_F, s_F$  are the same as in estimate (80). Integrating by parts, we see that

$$\begin{aligned} & \left| \sum_{\lambda_p \geq E} \exp(-\lambda_p t) \phi_p(x) \phi_p(\tilde{x}) \right| \\ & \leq C_1 \int_{\lambda \geq \alpha E} \exp(-\lambda t) [t(1 + \lambda^2)^{s_F} - 2s_F \lambda^{2s_F-1}] N_X(\lambda) d\lambda \\ & \leq C_2 (1 + t) \int_{\lambda \geq \alpha E} \exp(-\lambda t) (1 + \lambda^{n/2+2s_F}) d\lambda, \end{aligned} \tag{107}$$

where at the last stage we use (67). As

$$\exp(-\lambda t) \leq \exp(-(1 - \sigma)Et) \exp(-\sigma \lambda t/2), \quad \text{for } \lambda > \alpha E,$$

estimate (106) follows from the above inequality. QED

**Theorem 5.17** *For any  $\widehat{\varepsilon} > 0$ , there is  $\delta = \delta(\widehat{\varepsilon}) > 0$  such that, if  $X, X' \in \overline{\mathfrak{MM}}_p$  satisfy  $d_{pmGH}(X, X') < \delta$ , then there are  $\widehat{\varepsilon}$ -nets  $\{x_i\}_{i=1}^{I(\widehat{\varepsilon})} \subset X$  and  $\{x'_i\}_{i=1}^{I(\widehat{\varepsilon})} \subset X'$  satisfying (65) and*

$$|H_X(x_i, x_j, t) - H_{X'}(x'_i, x'_j, t)| < \widehat{\varepsilon}, \quad \text{for } t > \widehat{\varepsilon}. \tag{108}$$

**Proof.** By Proposition 5.16 with  $\sigma = 1/2$ , for sufficiently small  $\widehat{\varepsilon}$  and  $E = E(\widehat{\varepsilon}) = c\widehat{\varepsilon}^{-2}$ , the right-hand side of (106) is smaller than  $\widehat{\varepsilon}/4$  for any  $X \in \overline{\mathfrak{MM}}_p$ ,  $t > \widehat{\varepsilon}$ .

By (67), for any  $X \in \overline{\mathfrak{MM}}_p$  we can choose in the interval  $(-1, c\widehat{\varepsilon}^{-2} + 4\widehat{\varepsilon}^{\beta+n/2+1})$ , where  $\beta > 0$  is to be chosen later, the subintervals  $(a_\ell, b_\ell)$ ,  $\ell = 1, \dots, L = L(X)$ , with the following properties:

- (i)  $b_\ell - a_\ell \leq \widehat{\varepsilon}^\beta$ ,
- (ii)  $a_{\ell+1} > b_\ell + 4\widehat{\varepsilon}^{\beta+n+1}$ ,

- (iii)  $\text{spec}(-\Delta_X) \cap (-\infty, c\widehat{\varepsilon}^{-2}) \subset \cup_{\ell=1}^L (a_\ell, b_\ell),$
- (iv)  $d(\text{spec}(-\Delta_X), a_\ell), d(\text{spec}(-\Delta_X), b_\ell) > 4\widehat{\varepsilon}^{\beta+n+1}, \ell = 1, \dots, L.$

Note that this system of subintervals, associated with  $X$ , satisfies conditions of Definition 5.14 with  $d_\ell = 4\widehat{\varepsilon}^{\beta+n+1}$ . Denote it by  $\mathcal{I} = \mathcal{I}_{\widehat{\varepsilon}}(X)$  and consider  $\overline{\mathfrak{M}\mathfrak{M}}_{\mathcal{I}}$ . Choose  $d'_\ell = 3\widehat{\varepsilon}^{\beta+n+1}$  to define the corresponding  $\mathcal{J}(X)$ . Then there is  $\tau = \tau(\mathcal{I}(X), \widehat{\varepsilon}, \beta)$ , such that, if

$$X' \in \mathcal{U}_{\tau, X}(\overline{\mathfrak{M}\mathfrak{M}}_{\mathcal{I}}) = \{X' : d_{pmGH}(\overline{\mathfrak{M}\mathfrak{M}}_{\mathcal{I}}, X') < \tau\},$$

then  $X' \in \overline{\mathfrak{M}\mathfrak{M}}_{\mathcal{J}(X)}$ . Clearly,  $\{\mathcal{U}_{\tau, X} : X \in \overline{\mathfrak{M}\mathfrak{M}}_p\}$  form an open covering of  $\overline{\mathfrak{M}\mathfrak{M}}_p$ . Choose a finite subcovering,  $\mathcal{U}_1, \dots, \mathcal{U}_N$ ,  $N = N(\widehat{\varepsilon})$ . Observe that

$$\mathcal{U}_k \subset \overline{\mathfrak{M}\mathfrak{M}}_{\mathcal{J}(X)}, \quad \text{for some } X = X_k.$$

Then, due to Corollary 5.15, there is  $\sigma_k > 0$  so that if  $d_{pmGH}(X, X') < \sigma_k$ ,  $X \in \mathcal{U}_k$ , then equations (103), (104) are valid with  $\varepsilon = \widehat{\varepsilon}^n$  (and  $d''_\ell = 2\widehat{\varepsilon}^{\beta+n+1}$ ). Since

$$\bigcup_{k=1}^N \mathcal{U}_k = \overline{\mathfrak{M}\mathfrak{M}}_p,$$

taking  $\sigma(\widehat{\varepsilon}) = \min \sigma_k$ , we see that, for any  $X, X' \in \overline{\mathfrak{M}\mathfrak{M}}_p$ , if  $d_{pmGH}(X, X') < \sigma(\widehat{\varepsilon})$ , equations (103), (104) are valid with  $\varepsilon = \widehat{\varepsilon}^n$ . Also there is  $k$  such that  $X, X' \in \mathcal{U}_k$ .

Consider

$$\begin{aligned} & \left| \sum_{\lambda_p < c\widehat{\varepsilon}^{-2}} e^{-\lambda_p t} \phi_p(x) \phi_p(\tilde{x}) - \sum_{\lambda_p < c\widehat{\varepsilon}^{-2}} e^{-\lambda'_p t} \phi'_p(f(x)) \phi'_p(f(\tilde{x})) \right| \quad (109) \\ &= \left| \sum_{\ell=1}^L \left( \sum_{\lambda_p \in (a_\ell, b_\ell)} e^{-\lambda_p t} \phi_p(x) \phi_p(\tilde{x}) - \sum_{\lambda'_p \in (a_\ell, b_\ell)} e^{-\lambda'_p t} \phi'_p(f(x)) \phi'_p(f(\tilde{x})) \right) \right|, \end{aligned}$$

where  $f : X \rightarrow X'$ ,  $f' : X' \rightarrow X$  are almost inverse  $\sigma$ - approximations. We analyse each term in the right side of (109) separately.

First, observe that for  $t \leq \widehat{\varepsilon}^{-1}$ ,  $\lambda_p, \lambda'_p \in (a_\ell, b_\ell)$ ,

$$\max(|e^{-\lambda_p t} - e^{-a_\ell t}|, |e^{-\lambda'_p t} - e^{-a_\ell t}|) \leq c(\beta) \widehat{\varepsilon}^{(\beta-1)} e^{-a_\ell t}.$$



Using (80) together with (67), we see from the previous equation that replacing in the right side of (109) the exponents  $e^{-\lambda_p t}$  and  $e^{-\lambda_{\ell} t}$  by  $e^{-a_{\ell} t}$  gives rise to an error which can be estimated by  $C\widehat{\varepsilon}^{(\beta-1-n-2s_F)}$  for any  $t < \widehat{\varepsilon}^{-1}$ . Let us next use  $\beta = 3 + n + 2s_F$ . Then there is  $\widehat{\varepsilon}_1$  such that, for  $\widehat{\varepsilon} < \widehat{\varepsilon}_1$ ,

$$\left| \sum_{\ell=1}^L \sum_{\lambda_p \in (a_{\ell}, b_{\ell})} (e^{-\lambda_p t} - e^{-a_{\ell} t}) \phi_p(x) \phi_p(\tilde{x}) - \sum_{\ell=1}^L \sum_{\lambda'_p \in (a_{\ell}, b_{\ell})} (e^{-\lambda'_p t} - e^{-a_{\ell} t}) \phi'_p(f(x)) \phi'_p(f(\tilde{x})) \right| < \frac{\widehat{\varepsilon}}{16}.$$

Next, observe that

$$\sum_{i=1}^{n(\ell)} \left( \sum_{j=1}^{n(\ell)} u_{ij}^{\ell} f^*(\phi'_{j,\ell})(x) \right) \cdot \left( \sum_{k=1}^{n(\ell)} u_{ik}^{\ell} f^*(\phi'_{k,\ell})(\tilde{x}) \right) = \sum_{i=1}^{n(\ell)} f^*(\phi'_{i,\ell})(x) f^*(\phi'_{i,\ell})(\tilde{x}).$$

This equality, together with (80) and (64) with  $\varepsilon = \widehat{\varepsilon}^{\beta}$  implies that, that there is  $0 < \widehat{\varepsilon}_2 < \widehat{\varepsilon}_1$ , such that for  $\widehat{\varepsilon} < \widehat{\varepsilon}_2$

$$\begin{aligned} & |H_X(x, \tilde{x}, t) - H_{X'}(f(x), f(\tilde{x}), t)| \\ & \leq \frac{1}{2}\widehat{\varepsilon} + \sum_{\ell=1}^L \sum_{i=1}^{n(\ell)} e^{-a_{\ell} t} |\phi_{i,\ell}(x) \phi_{i,\ell}(\tilde{x}) - \phi_{i,\ell}^*(x) \phi_{i,\ell}^*(\tilde{x})|. \end{aligned}$$

Therefore, if equation (104) is valid with  $\varepsilon = \widehat{\varepsilon}^{\beta}$ , there is  $\widehat{\varepsilon}_3 < \widehat{\varepsilon}_2$  so that, for  $\widehat{\varepsilon} < \widehat{\varepsilon}_3$

$$|H_X(x, \tilde{x}, t) - H_{X'}(f(x), f(\tilde{x}), t)| < \widehat{\varepsilon}, \quad (110)$$

for  $\widehat{\varepsilon} < t < \widehat{\varepsilon}^{-1}$ .

Note also that  $\lambda_0 = \lambda'_0 = 0$ ,  $\phi_0(x) = \phi'_0(F(x)) = 1$  and, for  $p \geq 1$ ,  $\lambda_p \geq c > 0$ , where  $c$  is the same constant as in (67). Therefore, it follows from (80) that, there is  $0 < \widehat{\varepsilon}_4 < \widehat{\varepsilon}_3$  such that if  $\widehat{\varepsilon} < \widehat{\varepsilon}_4$ , then

$$\begin{aligned} & |H_X(x, \tilde{x}, t) - 1| < \widehat{\varepsilon}/2, \quad |H_{X'}(x', \tilde{x}', t) - 1| < \widehat{\varepsilon}/2, \\ & \text{for } t > \widehat{\varepsilon}^{-1}, \quad x, y \in X, \quad x', \tilde{x}' \in X'. \end{aligned}$$

This implies inequality (110) for all  $t > \widehat{\varepsilon}$ . Thus, we see that  $H_X, H_{X'}$  satisfy (110) for  $t > \widehat{\varepsilon}$ , if  $\delta < \delta_X = \sigma(\widehat{\varepsilon}^{3+n+2s_F})$ ,  $\widehat{\varepsilon} < \widehat{\varepsilon}_4$ .

At last, choosing  $\{x_i\}_{i=1}^{I(\widehat{\varepsilon})} \subset X$  to be an  $\widehat{\varepsilon}/2$ -net in  $X$ , we see from (64) that, for  $\delta < \delta(\widehat{\varepsilon})$ , the set  $\{f(x_i)\}_{i=1}^{I(\widehat{\varepsilon})} \subset X$  is  $\widehat{\varepsilon}$ -dense in  $X'$ . Due to the compactness of  $\overline{\mathfrak{MM}}_p$ ,  $I(\widehat{\varepsilon})$  can be chosen infirmly on  $\overline{\mathfrak{MM}}_p$ .

QED

Next we obtain an estimate for the continuity of the heat kernel:

**Corollary 5.18** *Let  $X \in \overline{\mathfrak{MM}}_p$  and  $t > \widehat{\varepsilon}$ . Then there is a  $a > 0$  such that for any  $x, x', \widetilde{x}, \widetilde{x}' \in X$ ,  $t, t' \in (\widehat{\varepsilon}, \widehat{\varepsilon}^{-1})$ , such that  $d_X(x, x') + d_X(\widetilde{x}, \widetilde{x}') + |t - t'| < a\widehat{\varepsilon}$ , we have*

$$\begin{aligned} & |H_X(x, \widetilde{x}, t) - H_X(x', \widetilde{x}', t')| \\ & < c(a)\widehat{\varepsilon}^{-(2s_F+n/2+2)} (d_X(x, x') + d_X(\widetilde{x}, \widetilde{x}') + |t - t'|). \end{aligned} \quad (111)$$

**Proof.** Similar to the proof of Proposition 5.16, we see that, up to an error of order  $\frac{1}{4}\widehat{\varepsilon}(d_X(x, x') + d_X(\widetilde{x}, \widetilde{x}') + |t - t'|)$ , we can approximate, for  $t > \widehat{\varepsilon}$ , the heat kernels  $H_X(x, \widetilde{x}, t)$ ,  $H_X(x', \widetilde{x}', t')$  by their eigenfunction expansion with  $\lambda < c\widehat{\varepsilon}^{-2}$ . Next, using (80) we obtain, similar to (107), that

$$\begin{aligned} & |H_X(x, \widetilde{x}, t) - H_X(x', \widetilde{x}', t')| (d_X(x, x') + d_X(\widetilde{x}, \widetilde{x}') + |t - t'|)^{-1} \\ & < \left( \frac{1}{4}\widehat{\varepsilon} + C(1+t) \int_{c < \lambda < c'\widehat{\varepsilon}^{-2}} e^{-\lambda t} [1 + \lambda^{2s_F+1}] N_X(\lambda) d\lambda \right). \end{aligned}$$

Using (67) and integrating the right-hand side, we obtain, for  $t > \widehat{\varepsilon}$ , estimate (111).

QED

## 6 From the local spectral data to the metric-measure structure

In this section we consider the inverse spectral problem on  $\overline{\mathfrak{MM}}_p$ .

**Definition 6.1** *Let  $\Omega \subset X \in \overline{\mathfrak{MM}}_p$  be open and non-empty. The set  $(\Omega, (\lambda_k)_{k=0}^\infty, (\phi_k|_\Omega)_{k=0}^\infty)$  is called the local spectral data (LSD) for  $X$ .*

*Moreover, if  $(\Omega, (\lambda_k)_{k=1}^\infty, (\phi_k|_\Omega)_{k=0}^\infty)$  and  $(\Omega', (\lambda'_k)_{k=0}^\infty, (\phi'_k|_{\Omega'})_{k=0}^\infty)$  are the LSD of  $(X, \mu)$  and  $(X', \mu')$ , correspondingly, we say that these data are equivalent in map  $\Psi_\Omega : \Omega \rightarrow \Omega'$ , if  $\Psi_\Omega$  is a homeomorphism,  $\lambda_k = \lambda'_k$ ,  $k = 0, 1, \dots$  and*

$$\Psi_\Omega^*(\phi'_k|_{\Omega'}) = \phi_k|_\Omega, \quad \text{for all } k = 0, 1, \dots$$

Our aim is to prove the following the uniqueness result for the inverse problem:

**Theorem 6.2** *Let  $(\Omega, (\lambda_k)_{k=0}^\infty, (\phi_k|_\Omega)_{k=0}^\infty)$  and  $(\Omega', (\lambda')_{k=0}^\infty, (\phi'|_{\Omega'})_{k=0}^\infty)$ , where  $\Omega \subset X^{reg}$ ,  $\Omega' \subset X'^{reg}$ , be the LSD of  $(X, \mu)$ ,  $(X', \mu') \in \overline{\mathfrak{MM}}_p$ , correspondingly, and assume that these data are equivalent in homeomorphism  $\Psi_\Omega : \Omega \rightarrow \Omega'$ . Then there exists a measure-preserving isometry,  $\Psi : X \rightarrow X'$ , such that  $\Psi|_\Omega = \Psi_\Omega$ , i.e.*

$$d_{X'}(\Psi(x), \Psi(\tilde{x})) = d_X(x, \tilde{x}), \quad x, \tilde{x} \in X, \quad \mu = \Psi^*(\mu').$$

Moreover,  $\Psi : X^{reg} \rightarrow X'^{reg}$  is a  $C_*^3$ -Riemannian isometry.

**Corollary 6.3** *Let  $(X, p, \mu_X)$ ,  $(X', p', \mu_{X'}) \in \overline{\mathfrak{MM}}_p$ . Let  $z_\alpha, z'_\alpha$ ,  $\alpha = 1, 2, \dots$ , be dense in  $\Omega = B(p, r)$ ,  $\Omega' = B(p', r)$ ,  $r > 0$ , correspondingly, while  $t_\ell$ ,  $\ell = 1, 2, \dots$  be dense in  $(0, \infty)$ . Assume that*

$$H_X(z_\alpha, z_\beta, t_\ell) = H_{X'}(z'_\alpha, z'_\beta, t_\ell), \quad \alpha, \beta, \ell = 1, 2, \dots$$

Then the conclusion of Theorem 6.2 remains valid and, in addition,  $\Psi_\Omega(p) = p'$ .

**Proof of Corollary.** By Lemma 4.7, the conditions of the corollary imply that  $\lambda_j = \lambda'_j$ ,  $\phi_j(z_\alpha) = \phi'_j(z'_\alpha)$ ,  $\alpha = 1, 2, \dots$ ,  $j = 0, 1, \dots$ , where the last equality is considered modulus an orthogonal transformation in  $\mathcal{L}(\lambda_j)$  and, without loss of generality we assume this transformation to be equal to identity. By closure in  $\mathbb{R}^\infty$  of the set  $\Phi(z_\alpha) = \Phi'(z'_\alpha)$ ,  $\alpha = 1, \dots$ , where

$$\Phi(x) = (\phi_j(x))_{j=0}^\infty, \quad \Phi'(x') = (\phi'_j(x'))_{j=0}^\infty \quad (112)$$

we obtain that the images  $\Phi(\overline{\Omega}) = \Phi'(\overline{\Omega'}) \subset \mathbb{R}^\infty$ . Since  $\overline{\Omega}, \overline{\Omega'}$  are compact, by Lemma 4.4, there exists  $J \subset \mathbb{Z}_+$ , such that  $\Phi_J : \Omega \rightarrow \mathbb{R}^J$ ,  $\Phi'_J : \Omega' \rightarrow \mathbb{R}^J$ , where  $\Phi_J(x) = (\phi_{i(1)}(x), \dots, \phi_{i(J)}(x))$ , and similar for  $\Phi'_J$ , are homomorphisms on their images. Then these images coincide and

$$\Psi|_\Omega = (\Phi'_J)^{-1} \circ \Phi_J : \Omega \rightarrow \Omega'$$

is a homomorphism. By Remark 4.9, we find  $d = \dim(X) = \dim(X')$  and, again by Lemma 4.4,

$$\Phi_J(\Omega \cap X^{reg}) = \Phi'_J(\Omega' \cap X'^{reg}).$$

Then

$$\Psi|_{\Omega} : \Omega \cap X^{reg} \rightarrow \Omega' \cap X'^{reg}$$

is a homeomorphism which satisfy conditions of Theorem 6.2, with  $\Omega \cap X^{reg}$ ,  $\Omega' \cap X'^{reg}$  in place of  $\Omega, \Omega'$ .

Using Theorem 6.2, we find  $d_X, d_{X'}$  on  $\Phi_J(\overline{\Omega}), \Phi_J(\overline{\Omega}')$ . Since  $\Omega = B(p, r)$ ,  $\Omega' = B(p', r)$ , we see that  $\Psi_{\Omega}(p) = p'$ .

QED

The proof of Theorem 6.2 is rather long and will consist of several parts.

## 6.1 Blagovestchenskii identity on $\overline{\mathfrak{M}\mathfrak{M}}_p$

For  $(X, \mu_X) \in \overline{\mathfrak{M}\mathfrak{M}}_p$  consider the following initial-boundary value problem, IBVP,

$$\begin{aligned} (\partial_t^2 - \Delta_X) u(x, t) &= H(x, t), \quad \text{in } X \times \mathbb{R}_+, \\ u|_{t=0} &= u_0, \quad \partial_t u|_{t=0} = u_1, \end{aligned} \quad (113)$$

with  $H \in L^2(\Omega \times \mathbb{R}_+)$ , where

$$L^2(W \times (0, T)) = \{H \in L^2(X \times (0, T)) : \text{supp}(H) \subset W \times (0, T)\},$$

$u_0 \in H^1(X)$ ,  $u_1 \in L^2(X)$ . We denote the unique solution of (113), when  $u_0 = u_1 = 0$ , by  $u^H(x, t)$ . Sometimes, we denote it by  $u^H(t) = u^H(\cdot, t)$ . Recall that,

$$u^H(x, t) = \sum_{j=0}^{\infty} u_j^H(t) \phi_j(x), \quad u_j^H(t) = (u^H(t), \phi_j)_{L^2(X, \mu)}. \quad (114)$$

**Lemma 6.4** *Assume the we are given LSD (6.1), that is,  $\Omega \subset X^{reg}$  and  $(\lambda_j)_{j=0}^{\infty}, (\phi_j|_{\Omega})_{j=0}^{\infty}$ , and  $H \in L^2(\Omega \times \mathbb{R}_+)$ . These data determine the Fourier coefficients  $u_j^H(t)$ ,  $t \in \mathbb{R}$  of  $u^H(t) \in L_{loc}^2(\Omega \times \mathbb{R}_+)$  up to the unknown multiplicative constant  $c_0 > 0$ .*

**Proof** By Lemma 4.6, LSD determine the metric  $h$  and the density  $\widehat{\rho}$ . We have

$$\begin{aligned} (\partial_t^2 + \lambda_j) u_j^H(t) &= (\partial_t^2 + \lambda_j) \int_X u^H(x, t) \phi_j(x) \rho(x) d\mu_h(x) \\ &= \frac{1}{c_0} (\partial_t^2 + \lambda_j) \int_X u^H(x, t) \phi_j(x) \widehat{\rho}(x) dV_h(x), \end{aligned}$$

where

$$c_0 = \widehat{c}V(X), \quad (115)$$

with  $\widehat{c}$  defined in (56). Continuing the above equality, we have

$$\begin{aligned} & (\partial_t^2 + \lambda_j) u_j^H(t) \\ &= \frac{1}{c_0} \int_X H(x, t), \phi_j(x) \widehat{\rho}(x) dV_h(x) + \frac{1}{c_0} \int_X (\Delta_X + \lambda_j) u^H(x, t) \phi_j(x) \widehat{\rho}(x) dV_h(x) \\ &= \frac{1}{c_0} \int_X H(x, t), \phi_j(x) \widehat{\rho}(x) dV_h(x) + \frac{1}{c_0} \int_X u^H(x, t) (\Delta_X + \lambda_j) \phi_j(x) \widehat{\rho}(x) dV_h(x) \\ &= \frac{1}{c_0} \int_\Omega H(x, t) \phi_j(x) \widehat{\rho}(x) dV_h(x). \end{aligned} \quad (116)$$

where we make use of (113). Since  $u_j^H(\cdot, 0) = \partial_t u_j^H(\cdot, 0) = 0$ , equations (116) imply that

$$u_j^H(t) = \frac{1}{c_0} \int_0^t \int_\Omega \frac{\sin(\sqrt{\lambda_j}(t-t'))}{\sqrt{\lambda_j}} H(x, t') \phi_j(x) \widehat{\rho}(x) dt' dV_h(x). \quad (117)$$

This equations shows that LSD determine  $c_0 u_j^H(t)$ ,  $j = 0, 1, \dots$ .

QED.

Let now  $(\Omega, (\lambda_k)_{k=1}^\infty, (\phi_k|_\Omega)_{k=1}^\infty)$  and  $(\Omega', (\lambda'_k)_{k=1}^\infty, (\phi'_k|_{\Omega'})_{k=1}^\infty)$  be LSD of the orbifolds  $X, X'$ , correspondingly, and assume that  $\lambda_k = \lambda'_k$ ,  $k = 1, 2, \dots$  and there is a homeomorphism  $\Psi_\Omega : \Omega \rightarrow \Omega'$  such that

$$\Psi_\Omega^*(\phi'_k|_{\Omega'}) = \phi_k|_\Omega, \quad k = 1, 2, \dots$$

Then the above Lemma implies that

$$c'_0 u_j^{H'}(t) = c_0 u_j^H(t), \quad \text{if } H' := (\Psi_\Omega^{-1})^*(H) \in L^2(\Omega' \times \mathbb{R}_+). \quad (118)$$

**Remark 6.5** Formula (117) implies that

$$U^H(x, t) \in C(\mathbb{R}_+, H^1(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)).$$

## 6.2 Approximate controllability

**Definition 6.6** *Let  $W \subset X$  be open,  $W \neq \emptyset$ . The domain of influence of  $W$  in  $X$  at time  $T$ ,  $X(W, T)$  is the set*

$$X(W, T) = \{x \in X : d(x, W) \leq T\}.$$

Return to IBVP (113). We have the following generalization of the classical result on the wave propagation.

**Lemma 6.7** *Let  $W \subset X$  be open. Assume  $\text{supp}(u_0), \text{supp}(u_1) \subset W$ ,  $\text{supp}(H) \subset W \times \mathbb{R}_+$ . Then, for any  $T \in \mathbb{R}$ ,*

$$\text{supp}(u(\cdot, T)) \subset X(W, T).$$

**Proof.** We start with IBVP on  $Y \in \overline{\mathfrak{FMM}}$ ,

$$\begin{aligned} (\partial_t^2 - \Delta_Y) u^*(y, t) &= H^*(y, t), \quad \text{in } Y \times \mathbb{R}_+, \\ u^*|_{t=0} &= u_0^*(y), \quad \partial_t u^*|_{t=0} = u_1^*(y). \end{aligned}$$

Since  $\rho^*, h^* \in C_*^2(Y)$ , a slight modification of the classical results shows that

$$\text{supp}(u^*(\cdot, T)) \subset Y(W^*, T),$$

where  $W^* \subset Y$  and  $Y(W^*, T)$  is the corresponding domain of influence.

Observe that, using the Fourier decomposition, if  $u_0^*, u_1^*, H^*$  are  $O(n)$ -invariant, then, for any  $t$ ,  $u^*(t)$  is also  $O(n)$ -invariant. Thus, if  $u_0^* = \pi^*(u_0)$ ,  $u_1^* = \pi^*(u_1)$  and  $H^* = \pi^*(H)$ , by (239), we have  $u(t) = \pi_* u^*(t)$ . Let  $W^* = \pi^{-1}(W)$ . Then, by (241), for any  $T \in \mathbb{R}_+$ ,

$$Y(W^*, T) = \pi^{-1}(X(W, T)),$$

so that  $u(T) = 0$  outside  $X(W, T)$ .

QED

Next we prove Tataru's controllability result.

**Theorem 6.8** *Let  $W \subset X$  be an open set. Then the set  $\{u^H(\cdot, T) : H \in C_*^3(X \times (0, T))\}$ ,  $\text{supp}(H) \subset W \times (0, T)$ , is dense in  $L^2(X(W, T))$ .*

**Proof** Let first  $W^*$  be an open subset of  $Y$  and consider the set  $\{(u^*)^{H^*}(\cdot, T) : H^* \in C_*^3(Y \times (0, T))\}$ ,  $\text{supp}(H^*) \subset W^* \times (0, T)$ . Since  $\rho^*, h^* \in C_*^2(Y)$ , the classical Tataru's unique continuation result, see [60], remains valid for this case. Using the standard duality arguments, see e.g. [41], sec. 2.5, this implies that, for any  $a^* \in L^2(Y)$ ,  $\text{supp}(a^*) \subset Y(W^*, T)$  and  $\varepsilon > 0$ , there is  $H^*(a^*, \varepsilon) \in C_*^3(W^* \times (0, T))$ ,  $\text{supp}(H^*) \subset W^* \times (0, T)$ , such that

$$\|a^* - (u^*)^{H^*}(T)\|_{L^2(Y)} < \varepsilon. \quad (119)$$

Let now  $W^*$  and  $a^*$  be  $O(n)$ -invariant, i.e.  $W^* = \pi^{-1}(W)$ ,  $W \subset X$ , and  $a^* = \pi^*(a)$ ,  $\text{supp}(a) \subset W$ . Denote

$$H_O(a, \varepsilon) = \mathbb{P}_O(H^*(a^*, \varepsilon)), \quad \text{supp}(H_O) \subset W^* \times (0, T),$$

where the last inclusion follows from (44) together with  $O(n)$ -invariance of  $W^*$ . In addition, see considerations following (44),  $H_O(a, \varepsilon) \in C_*^3(W^* \times (0, T))$ .

By (237) we then have

$$\mathbb{P}_O((u^*)^{H^*}(T)) = (u^*)^{H_O}(T),$$

which, together with (119) and  $a^* \in L_O^2$ , imply that

$$\|a^* - (u^*)^{H_O}(T)\|_{L^2(Y)} < \varepsilon. \quad (120)$$

Let  $H(a, \varepsilon) = \pi_*(H_O(a, \varepsilon))$ . Then, using again (239), we see that

$$(u^*)^{H_O}(T) = \pi^*(u^H(T)).$$

This, together with (120) and  $a^* = \pi^*(a)$ , shows that  $\|a - u^{H(a, \varepsilon)}(T)\| < \varepsilon$ .

QED

**Lemma 6.9** *For any open  $W \subset \Omega$  and  $s > 0$ , there exist  $\{F_k\}_{k=1}^\infty \subset L^2(W \times (0, s))$  such that for  $a_k(x) = u^{F_k}(x, s)$ ,*

$$(*) \quad \{a_k(x)\}_{k=1}^\infty \text{ form an orthonormal basis in } L^2(X(W, s), c_0^2 \mu_X).$$

*Furthermore, using the LSD, that is,  $\Omega, \lambda_j, \phi_j|_\Omega, j = 0, 1, \dots$  it is possible to construct sources  $F_k$  so that  $(*)$  holds.*

**Proof** Let  $H_k \in C_*^3(W \times (0, s))$ ,  $k = 1, 2, \dots$ , be dense in  $L^2(W \times (0, s))$ . Then, by Theorem 6.8,

$$\overline{\{b_k(x) : k = 1, 2, \dots\}} = L^2(X(W, s), \mu_X), \quad b_k(x) := u^{H_k}(x, s). \quad (121)$$

By Lemma 6.4,  $\Omega, \lambda_j, \phi_j|_\Omega$ ,  $j = 0, 1, \dots$ , determine  $\langle\langle b_k, b_l \rangle\rangle$ , where we denote

$$\langle\langle b_k, b_l \rangle\rangle = c_0^2 \sum_{j=0}^{\infty} u_j^{F_k}(s) u_j^{F_l}(s) = c_0^2 (b_k, b_l)_{L^2(X, \mu_X)}. \quad (122)$$

By applying the Gram-Smidt orthonormalization algorithm to the sequence  $(b_k)_{k=1}^{\infty}$  with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ , we obtain a sequence  $(a_k)_{k=1}^{\infty}$ , where  $a_k = \sum_{l=1}^{l(k)} c_{kl} b_l$  such that  $\langle\langle a_k, a_l \rangle\rangle = \delta_{kl}$ . Note that, for  $\hat{a}_k = c_0 a_k$ , we have  $(\hat{a}_k, \hat{a}_l)_{L^2(X)} = \delta_{kl}$ . Then the claim holds for

$$a_k(x) = u^{F_k}(s), \quad F_k(x, t) = \sum_{l=1}^{l(k)} c_{kl} H_l(x, t).$$

QED

By Lemma 6.9, we have

$$\chi_{X(W, s)}(x) v(x) = \sum_{k=1}^{\infty} (v, \hat{a}_k) \hat{a}_k(x) = \sum_{k=1}^{\infty} \langle\langle v, a_k \rangle\rangle a_k(x),$$

where  $\chi_A$  is the characteristic function of  $A \subset X$ . Assume now that we are given a sequence  $(p_j)_{j=0}^{\infty} \in \ell^2$  and denote

$$v(x) = \sum_{j=0}^{\infty} p_j \phi_j(x) \in L^2(X).$$

Let  $(q_j)_{j=0}^{\infty} \in \ell^2$  be such that

$$\sum_{j=0}^{\infty} q_j \phi_j(x) = \chi_{X(W, s)}(x) v(x).$$



Combining the above formulas, we see that

$$\begin{aligned}
q_j &= (\chi_{X(W,s)} v, \phi_j) = \sum_{k=1}^{\infty} (v, \widehat{a}_k) (\widehat{a}_k, \phi_j) \\
&= c_0^{-2} \sum_{k=1}^{\infty} \langle \langle v, a_k \rangle \rangle \langle \langle a_k, \phi_j \rangle \rangle \\
&= c_0^{-2} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} p_j \langle \langle \phi_j, u^{F_k}(s) \rangle \rangle \langle \langle u^{F_k}(s), \phi_k \rangle \rangle \\
&= \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} p_{\ell} (c_0(\phi_{\ell}, u^{F_k}(s))) (c_0(u^{F_k}(s), \phi_j)).
\end{aligned} \tag{123}$$

Thus, as we can compute  $(c_0(u^{F_k}(s), \phi_j))$  using Lemma 6.4, we see that, if we are given LSD and a sequence  $(p_j)_{j=1}^{\infty} \in \ell^2$ , we can determine sequence  $(q_j)_{j=1}^{\infty} \in \ell^2$  so (123) holds.

Since formula (123) determines the Fourier coefficients of  $\chi_{X(W,s)} v$ , we see that, if we are given LSD, then we can construct the orthogonal projector  $M_{(W,s)}$ ,

$$M_{(W,s)} : \ell^2 \rightarrow \ell^2, \quad M_{(W,s)}((p_j)_{j=0}^{\infty}) = (\mathcal{F} \circ \chi_{X(W,s)} \circ \mathcal{F}^{-1})((p_j)_{j=0}^{\infty}), \tag{124}$$

where  $\mathcal{F} : v \mapsto (\int_{\Omega} v(x) \phi_j(x) d\mu_X)_{j=0}^{\infty}$  is the Fourier transform.

These considerations imply

**Lemma 6.10** *Let  $(\Omega, (\lambda_j)_{j=0}^{\infty}, (\phi_j|_{\Omega})_{j=0}^{\infty})$  and  $(\Omega', (\lambda'_j)_{j=0}^{\infty}, (\phi'_j|_{\Omega'})_{j=0}^{\infty})$  be the LSD associated to  $(X, \mu)$  and  $(X', \mu')$  and assume that these data are equivalent in homeomorphism  $\Phi_{\Omega} : \Omega \rightarrow \Omega'$ .*

*Let  $W \subset \Omega$  be open,  $s > 0$ , and  $W' = \Phi_{\Omega}(W)$ . Then the maps (124) corresponding to  $(X, \mu)$  and  $(X', \mu')$  and sets  $W$  and  $W'$  coincide,*

$$M_{(W,s)} = M_{(W',s)}. \tag{125}$$

*In particular, for any  $j, l \in \mathbb{Z}_+$ ,*

$$\int_{X(W,s)} \phi_j(x) \phi_l(x) d\mu_X = \int_{X'(W',s)} \phi'_j(x') \phi'_l(x') d\mu_{X'}. \tag{126}$$

**Proof.** Since (125) is proven, let us show (126). Note that (125) implies that

$$\mathcal{F} \circ \chi_{X(W,s)} \circ \mathcal{F}^{-1}(\ell^2) = \mathcal{F}' \circ \chi_{X'(W',s)} \circ \mathcal{F}'^{-1}(\ell^2).$$

As  $\mathcal{F}(\phi_j) = \mathcal{F}'(\phi'_j) = e_j$ , where  $e_j = (0, \dots, 0, 1, 0, \dots)$ , with the only 1 on the  $j$ -th place, (126) follows.

QED

We apply the previous lemma to find inner products of eigenfunctions over small sets.

**Lemma 6.11** *Let the assumption of Lemma 6.10 be valid and let  $W_l \subset \Omega$  be open sets and  $s_l \geq 0$ ,  $l = 1, \dots, 2p$ , and  $W'_l := \Phi_\Omega(W_l)$  be satisfying for  $1 \leq l \leq p$ ,*

$$X(W_{2l-1}, s_{2l-1}) \supset X(W_{2l}, s_{2l}), \quad X(W'_{2l-1}, s_{2l-1}) \supset X(W'_{2l}, s_{2l}),$$

where some  $W_{2l}$  may be empty. Denote

$$I := \bigcap_{l=1}^p (X(W_{2l-1}, s_{2l-1}) \setminus X(W_{2l}, s_{2l})), \quad (127)$$

$$I' := \bigcap_{l=1}^p (X(W'_{2l-1}, s_{2l-1}) \setminus X(W'_{2l}, s_{2l})), \quad (128)$$

Then for any  $j, l \in \mathbb{Z}_+$ ,

$$\int_I \phi_j(x) \phi_l(x) d\mu_X = \int_{I'} \phi'_j(x') \phi'_l(x') d\mu_{X'}. \quad (129)$$

**Proof.** The result follows directly from Lemma 6.10.

QED

### 6.3 Cut locus

We use the fact that , if  $X \in \overline{\mathfrak{MM}}$ , then it is a length space and that all locally compact length spaces are geodesics spaces, that is, all points can be joined with a shortest, i.e. length minimizing path. This follows from

the fact that a GH limit of a sequence of locally compact length space is again a locally compact length spaces, see [36]. Another way to see this is that, since  $Y \in \overline{\mathfrak{FMM}}$  satisfies  $h_Y \in C_*^2$ , the classical Hopf-Rinow theorem remains valid, and there is a shortest geodesic between any  $y, y' \in Y$ . Since  $X = \pi(Y) = Y/O(n)$ , the result for  $\overline{\mathfrak{MM}}$  follows from (241).

**Definition 6.12** *We say that a path  $\gamma : [0, \ell] \rightarrow X$  is a geodesic on  $X$  if it is the locally distance minimizing path and is parametrized along the arc length.*

*In particular, if  $x \in X^{reg}$ ,  $\xi \in S_x(X)$ , we denote by  $\gamma_{x,\xi}(t)$ ,  $t \geq 0$  a geodesic of  $X$  which, for  $t > 0$  small enough, coincides with a Riemannian geodesic starting from  $x$  in the direction  $\xi$ . We denote the largest interval  $[0, T]$  on which the geodesic  $\gamma_{x,\xi}(t)$  can be defined by  $I_{x,\xi}$ .*

Note that if  $\gamma$  is a geodesic of  $X$ , then  $\gamma \cap X^{reg}$  is a geodesic in the sense of the Riemannian geometry.

**Definition 6.13** For  $x \in X^{reg}$  and  $\xi \in S_x(X)$ , let  $i(x, \xi)$  be the supremum of those  $t \in I_{x,\xi}$  that  $\gamma_{x,\xi}$  is defined and a distance minimizing path between  $\gamma_{x,\xi}(0)$  and  $\gamma_{x,\xi}(t)$ . Then  $i(x, \xi)$  is called the cut locus distance function on  $S(X^{reg})$ .

Now the injectivity radius  $i(x)$  at  $x$  is defined as

$$i(x) = \inf_{\xi \in S_x(X)} i(x, \xi).$$

Note that  $i(x) > 0$  for any  $x \in X^{reg}$  due to the boundedness of the sectional curvature from above.

**Lemma 6.14** *For  $x \in X^{reg}$  we have*

$$i(x) \leq d_X(x, X^{sing}). \quad (130)$$

**Proof.** Assume the opposite and let  $x' \in X^{sing}$  be the nearest point to  $x$ . Then,

$$t_0 = d_X(x, x') < i(x) \quad (131)$$

and there exists a shortest  $\gamma_{x,\xi}(t)$  connecting  $x$  and  $x'$ ,  $\gamma_{x,\xi}(t_0) = x'$ . Inequality (131) implies that  $\gamma_{x,\xi}(t)$  remains to be length minimizing at least till  $t = i(x)$ . However, due to [53], this yields that  $\gamma_{x,\xi}(t) \in X^{reg}$ , if  $0 \leq t < i(x)$ , contradicting  $\gamma_{x,\xi}(t_0) \in X^{sing}$ .

QED

Let  $x_1 = \gamma_{x,\xi}(\rho)$  with  $0 < \rho < i(x)$ ,  $s, \varepsilon > 0$ , and denote

$$N(x, \xi; s, \rho, \varepsilon) = N(s, \rho, \varepsilon) = (B_{\rho+s}(x) \cup B_{s+\varepsilon}(x_1)) \setminus B_{\rho+s}(x). \quad (132)$$

Clearly,  $N(s, \rho, \varepsilon_2) \subset N(s, \rho, \varepsilon_1)$ , if  $\varepsilon_2 < \varepsilon_1$ , and  $B_{\rho+s}(x) \cup B_{s+\varepsilon}(x_1) = X(B_\rho(x) \cup B_\varepsilon(x_1), s)$ .

**Lemma 6.15** *Let  $x \in X^{reg}$  and suppose that  $\rho < i(x)$ .*

(a) *If  $s + \rho < i(x)$ , then*

$$\bigcap_{\varepsilon > 0} \overline{N}(x, \xi; s, \rho, \varepsilon) = \{\gamma_{x,\xi}(\rho + s)\}.$$

(b) *If  $s + \rho > i(x)$ , then there are  $\xi \in S_x(X)$  and  $\varepsilon > 0$  such that*

$$N(x, \xi; s, \rho, \varepsilon) = \emptyset.$$

(c) *The injectivity radius satisfies*

$$i(x) = \inf_{s > 0} \{s + \rho; \text{ there are } \xi \in S_x X \text{ and } \varepsilon > 0 \text{ such that } N(x, \xi; s, \rho, \varepsilon) = \emptyset\}.$$

**Proof of Lemma 6.15** (a) For any  $\xi \in S_x(X)$  and any sufficiently small  $\varepsilon$ ,  $\gamma_{x,\xi} : [0, s + \rho + \varepsilon] \rightarrow X$  is length minimizing. This implies that

$$\gamma_{x,\xi}([s + \rho, s + \rho + \varepsilon]) \subset \overline{N}(x, \xi; s, \rho, \varepsilon).$$

Thus,  $\gamma_{x,\xi}(s + \rho) \in \bigcap_{\varepsilon} \overline{N}(x, \xi; s, \rho, \varepsilon)$ .

(b) Take  $\xi \in S_x(X)$  with  $i(x, \xi) < \rho + s$ . Let us then show that there is  $\delta > 0$  so that  $d_X(x_1, \partial B_{\rho+s}(x)) > s + \delta$ . Assuming the opposite, there is  $x' \in \partial B_{\rho+s}(x)$  with  $d_X(x_1, x') = s$ . Denote by  $\mu(t)$  a distance minimizing path from  $x_1$  to  $x'$ ,  $\mu(0) = x_1$ ,  $\mu(s) = x'$ .

Consider a path  $\gamma(t)$  which coincides with  $\gamma_{x,\xi}(t)$  for  $0 \leq t \leq \rho$ , and  $\gamma(t) = \mu(t - \rho)$ , for  $\rho \leq t \leq \rho + s$ . This path is distance minimizing until  $t = \rho + s$ . Indeed, this is certainly the case for  $t \leq \rho$ , as  $\rho < i(x)$ . Assume that, for some  $\rho < t_0 \leq \rho + s$ , there is another path  $\gamma'(t)$ ,  $\gamma'(0) = x$ , such that

$$\gamma'(t') = \gamma(t_0), \quad t' < t_0.$$

Since  $d_X(\gamma(t_0), x') = d_X(\mu(t_0 - \rho), \mu(s)) = s + \rho - t_0$ , we have

$$d_X(x, x') \leq d_X(x, \gamma(t_0)) + d_X(\gamma(t_0), x') \leq t' + s + \rho - t_0 < s + \rho.$$

This contradicts  $x' \in \partial B_{\rho+s}(x)$ . However, since  $\gamma$  is distance minimizing, in particular, it is the minimizing continuation, for  $\rho < t < \rho + s$ , of  $\gamma_{x,\xi}$  onto  $t \in (0, \rho + s)$ . This contradicts  $i_{x,\xi} < \rho + s$ .

Therefore, for  $\varepsilon < \delta$ , we have  $\overline{B}_{s+\varepsilon}(x_1) \subset B_{\rho+s}(x)$ , i.e.  $N(x, \xi; s, \rho, \varepsilon) = \emptyset$ .

As (a) and (b) are proven, (c) follows immediately by the definition of  $i(x)$ . QED.

Next we combine Lemma 6.11 and Lemma 6.15.

**Lemma 6.16** *Let  $(X, \mu_X), (X', \mu_{X'}) \in \overline{\mathfrak{MM}}$ . Assume  $(\Omega, (\lambda_k)_{k=0}^\infty, (\phi_k|_\Omega)_{k=0}^\infty)$  and  $(\Omega', (\lambda'_k)_{k=0}^\infty, (\phi'_k|_{\Omega'})_{k=0}^\infty)$ , where  $\Omega \in X^{reg}$ ,  $\Omega' \in X'^{reg}$  be the LSD of  $X, X'$ , correspondingly. Assume that these data are equivalent in homeomorphism  $\Phi_\Omega : \Omega \rightarrow \Omega'$ . Then  $\Phi_\Omega : \Omega \rightarrow \Omega'$  is a  $C_*^3$ -diffeomorphism.*

*Moreover, let  $x \in \Omega$  and  $x' := \Phi_\Omega(x)$  and  $\exp_x : T_x X \rightarrow X$  and  $\exp_{x'} : T_{x'} X' \rightarrow X'$  denote exponential maps. Then,  $i(x) = i(x')$  and, for  $r = i(x)$ , the map  $E : B(x, r) \rightarrow B'(x', r)$ ,*

$$E(z) = \exp_{x'} \left( d\Phi_\Omega|_x(\exp_x^{-1}(z)) \right)$$

*satisfies  $h = E^*(h')$ ,  $\widehat{\rho} = E^*(\widehat{\rho}')$ , and*

$$\phi_j(z) = \phi'_j(E(z)), \quad \text{for all } j = 0, 1, \dots, \text{ and } z \in B(z, r). \quad (133)$$

**Proof.** By Lemma 4.4, any  $z \in \Omega$  has a neighborhood  $V \subset \Omega$  such that there is an index  $\mathbf{j} = (j_1, \dots, j_d)$ , for which  $\Phi_{\mathbf{j}} : V \rightarrow \mathbb{R}^d$ ,  $d = \dim(X)$ ,  $\Phi_{\mathbf{j}}(x) = (\phi_j(x))_{j \in \mathbf{j}}$ , defines  $C_*^3$ -smooth coordinates. Let  $z' = \Phi_\Omega(z)$  and  $V' = \Phi_\Omega(V)$ . Then, as  $\phi'_j(\Phi|_\Omega(x)) = \phi_j(x)$  for all  $j$ , we see by Lemma 4.5

and Remark 4.9 that, for the same index  $\mathbf{j}$ , the map  $\Phi'_{\mathbf{j}} : V' \rightarrow \mathbb{R}^d$  defines  $C^3_*$ -smooth coordinates and moreover, that  $\Phi_{\Omega} = (\Phi'_{\mathbf{j}})^{-1} \circ \Phi_{\mathbf{j}}$  in  $V$ . Thus,  $\Phi_{\Omega} : \Omega \rightarrow \Omega'$  is a  $C^3_*$ -smooth diffeomorphism, and its differential  $d\Phi_{\Omega}|_y$  is well-defined.

Let now  $\rho < \min(i(x), i'(x'))$  be so small that  $B_{\rho}(x) \subset \Omega$ ,  $B_{\rho}(x') \subset \Omega'$ , and  $\xi \in S_x(X)$ ,  $\xi' := d\Phi_{\Omega}|_x(\xi) \in S_{x'}(X')$ . Take  $s > 0, \varepsilon > 0$ , and consider Lemma 6.11 with  $I = N(x, \xi; s, \rho, \varepsilon)$ ,  $I' = N(x', \xi'; s, \rho, \varepsilon)$  and  $j = \ell = 0$ . Since  $\phi_0 = 1$ ,  $\phi'_0 = 1$ , using equation (129) we see that

$$\mu_X(N(x, \xi; s, \rho, \varepsilon)) = \mu'_{X'}(N(x', \xi'; s, \rho, \varepsilon)).$$

Therefore, due to Lemma 6.15,  $i(x) = i(x')$ . Note also, that utilizing Lemmata 6.4 and 6.9, we can evaluate those volumes from LSD up to multiplication by the same unknown constant  $c_0$ .

Next, we again take  $\xi \in S_x(X)$ ,  $\xi' := d\Phi_{\Omega}|_x(\xi)$  and  $0 < s < i(x) - \rho$ . Consider  $x_0 = \gamma_{x, \xi}(\rho + s)$ ,  $x'_0 = \gamma'_{x', \xi'}(\rho + s) = E(x_0)$ . Then, by (129), we have for all  $j$

$$\begin{aligned} \frac{\phi_j(x_0)}{\phi_0(x_0)} &= \lim_{\varepsilon \rightarrow 0} \frac{\int_{N(x, \xi; \rho, s, \varepsilon)} \phi_j(z) \phi_0(z) \mu_X(z)}{\int_{N(x, \xi; \rho, s, \varepsilon)} \phi_0^2(z) \mu_X(z)} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_{N'(x', \xi'; \rho, s, \varepsilon)} \phi'_j(z') \phi'_0(z') \mu_{X'}(z')}{\int_{N'(x', \xi'; \rho, s, \varepsilon)} (\phi'_0)^2(z') \mu_{X'}(z')} = \frac{\phi'_j(x'_0)}{\phi'_0(x'_0)}. \end{aligned}$$

As  $\phi_0 = \phi'_0 = 1$ , this yields (133).

At last, using Lemma 4.6, we obtain that  $h_X = E^*(h_{X'})$ ,  $\widehat{\rho} = E^*(\widehat{\rho}')$ .

QED

**Proof of Theorem 6.2** (1) Let  $\mathcal{O}$  be the set of all pairs,  $(\Omega_1, \Omega'_1)$ , of connected open subsets  $\Omega_1 \subset X^{reg}$ ,  $\Omega'_1 \subset X'^{reg}$  containing  $(\Omega, \Omega')$  such that  $\Psi_{\Omega}$  extends to a map  $\Psi_1 : \Omega_1 \rightarrow \Omega'_1$  which is a local diffeomorphism and satisfies  $\Psi_1^*(h'|_{\Omega'_1}) = h|_{\Omega_1}$ ,  $\Psi_1^*(\widehat{\rho}'|_{\Omega'_1}) = \widehat{\rho}|_{\Omega_1}$ , and  $\Psi_1^*(\phi'_k|_{\Omega'_1}) = \phi_k|_{\Omega_1}$  for every  $k \in \mathbb{Z}_+$ . Note that by the last condition, the map  $\Psi_1$  has to coincide with  $\Phi' \circ \Phi^{-1}|_{\Omega_1}$ , where  $\Phi : X \rightarrow \mathbb{R}^{\mathbb{Z}_+}$  and  $\Phi' : X' \rightarrow \mathbb{R}^{\mathbb{Z}_+}$  are the maps defined in (112).

Let us consider  $\mathcal{O}$  as a partially ordered set, where the order is given by the inclusion  $\subset$ . If  $J$  is an ordered index set and  $(\Omega_j, \Omega'_j) \in \mathcal{O}$ ,  $j \in J$ , are sets so that  $\Omega_j \subset \Omega_k$ ,  $\Omega'_j \subset \Omega'_k$ , if  $j < k$ , and  $\Psi_j : \Omega_j \rightarrow \Omega'_j$  are the corresponding maps, it follows from Corollary 4.4 that  $\Psi_k|_{\Omega_j} = \Psi_j$ . Using this we see that

$(\bigcup_{j \in J} \Omega_j, \bigcup_{j \in J} \Omega'_j) \in \mathcal{O}$ . By Zorn Lemma, this yields that there exists a maximal element  $(\Omega_{\max}, \Omega'_{\max})$  in  $\mathcal{O}$ . Let  $\Psi_{\max} : \Omega_{\max} \rightarrow \Omega'_{\max}$  be the map corresponding to  $(\Omega_{\max}, \Omega'_{\max}) \in \mathcal{O}$ .

**Lemma 6.17** *The maximal element satisfies  $(\Omega_{\max}, \Omega'_{\max}) = (X^{reg}, (X')^{reg})$ .*

**Proof.** Let  $\Psi_{\max} : \Omega_{\max} \rightarrow \Omega'_{\max}$  be a local diffeomorphism corresponding to the set  $(\Omega_{\max}, \Omega'_{\max}) \in \mathcal{O}$ .

Assume that the claim is not valid. Then, without loss of generality, we can assume that  $X^{reg} \setminus \Omega_{\max} \neq \emptyset$ . As  $X^{reg}$  is connected, we see that there is point  $z_0$  in  $\partial\Omega_{\max} \cap X^{reg}$ . Let us choose a sequence of points  $y_k \in \Omega_{\max}$  such that  $y_k \rightarrow z_0$  as  $k \rightarrow \infty$  and let  $y'_k = \Psi_{\max}(y_k)$ . As  $X'$  is compact, by choosing a subsequence, we can assume that there exists  $\lim_{k \rightarrow \infty} y'_k = z'_0$ . Note that at this moment we do not know whether  $z'_0 \in (X')^{reg}$ .

Let  $W = B(z_0, t_0) \subset X^{reg}$  be a neighborhood of  $z_0$  such that  $\overline{W} \subset X^{reg}$ . By Lemma 2.2, there exists a positive number  $i_0 > 0$  such that  $\text{inj}(p) \geq i_0$  for all  $p \in \overline{W}$ . Let  $r = \min(i_0, t_0)/2$ .

Let us choose  $k$  such that  $z_0 \in B(y_k, r)$ . Since  $y_k \in \Omega_{\max} \subset X^{reg}$ ,  $y'_k \in \Omega'_{\max} \subset (X')^{reg}$ , there is  $\delta > 0$  such that  $\delta < \min(i(y_k), i(y'_k))$  and  $B = B(y_k, \delta) \subset \Omega_{\max}$ ,  $B' = B(y'_k, \delta) \subset \Omega'_{\max}$ . Then the LSD  $(B, \{\lambda_k\}_{k=0}^\infty, \{\phi_k|_B\}_{k=0}^\infty)$  and  $(B', \{\lambda'_k\}_{k=0}^\infty, \{\phi'_k|_{B'}\}_{k=0}^\infty)$  are equivalent by the map  $\Psi_{\max}|_B$ . Using Lemma 6.16, we see that  $i(y'_k) = i(y_k)$ , and moreover, as  $r \leq i(y)/2$ , there is a diffeomorphism  $E : B(y_k, 2r) \rightarrow B'(y'_k, 2r)$  such that

$$\phi_j(x) = \phi'_j(E(x)), \quad \text{for all } j = 0, 1, \dots, \text{ and } x \in B(y_k, 2r). \quad (134)$$

This implies that  $E$  coincides with the map  $\Phi' \circ \Phi^{-1}|_{B(y_k, 2r)}$ , where  $\Phi, \Phi'$  are defined in (112). But also  $\Psi_{\max}$  is the restriction of  $\Phi' \circ \Phi^{-1}$  to  $\Omega_{\max}$ . Since  $E : B(y_k, 2r) \rightarrow B'(y'_k, 2r)$  and  $\Psi_{\max} : \Omega_{\max} \rightarrow \Omega'_{\max}$  are surjections, we see that  $\Psi_{ext} = \Phi' \circ \Phi^{-1}|_{\Omega_{\max} \cup B(y_k, 2r)}$  is a bijection  $\Omega_{ext} = \Omega_{\max} \cup B(y_k, 2r) \rightarrow \Omega'_{ext} = \Omega'_{\max} \cup B'(y'_k, 2r)$ . In particular, we have  $\Psi_{ext}^*(\phi'_k) = \phi_k$ , and using Corollary 4.6, we see that  $\Psi_{ext}^*(h') = h$  and  $\Psi_{ext}^*(\hat{\rho}') = \hat{\rho}$  on  $\Omega_{ext}$ . Thus,  $(\Omega_{ext}, \Omega'_{ext}) \in \mathcal{O}$ . This implies, in particular, that  $B(z_0, r) \subset B(y_k, 2r) \subset X^{reg}$  and  $B'(z'_0, r) \subset B'(y'_k, 2r) \subset (X')^{reg}$ .

As  $z_0 \in B(y_k, 2r) \setminus \Omega_{\max}$ ,  $z'_0 \in B(y'_k, 2r) \setminus \Omega'_{\max}$ , we see that  $\Omega_{ext} \not\subset \Omega_{\max}$ ,  $\Omega'_{ext} \not\subset \Omega'_{\max}$ , which is in contradiction with the assumption that  $(\Omega_{\max}, \Omega'_{\max})$  is the maximal element of  $\mathcal{O}$ . Thus,  $(\Omega_{\max}, \Omega'_{\max}) = (X^{reg}, (X')^{reg})$ .

QED

**End of Proof of Theorem 6.2.** Since  $\overline{X^{reg}} = X$ ,  $\overline{(X')^{reg}} = X'$  and  $\Psi_{max}$  is an isometry between  $X^{reg}$  and  $(X')^{reg}$ , extending it by continuity to  $X$  we obtain an isometry  $\Psi : X \rightarrow X'$ . Then the same considerations as in Lemma 6.16, with  $\Omega = X^{reg}$ , show that  $\Psi|_{X^{reg}}$  is  $C_*^3$ - Riemannian isometry. Moreover, as

$$\mu_X(X^{sing}) = \mu_{X'}((X')^{sing}) = 0,$$

considerations of Lemma 6.16 show that  $\hat{\rho} = \Psi^*(\hat{\rho}')$ . Together with  $h = \Psi^*(h')$  and  $\mu_X = \mu_{X'} = 1$ , this yields that  $\rho = \Psi^*(\rho')$ .

QED

## 7 Stability of inverse problem

In this section we prove the main Theorem 1.2. Actually, we prove a slightly more general variant of this theorem which deals with arbitrary  $X, X' \in \overline{\mathfrak{MM}}_p$  rather than  $M, M' \in \mathfrak{MM}_p$ .

**Theorem 7.1** *Let  $\mathfrak{MM}_p$  be a class of pointed Riemannian manifolds  $(M, p; \mu_M)$  defined by conditions (6), (18), i.e. having dimension  $n$  with sectional curvature bounded by  $\Lambda$  (from above and below) and diameter. Let  $\overline{\mathfrak{MM}}_p$  be the closure of  $\mathfrak{MM}_p$  with respect to the pointed measured GH topology.*

*Then, for any  $r > 0$ , there exists a continuous increasing function*

$$\omega_r : [0, 1) \rightarrow [0, \infty), \quad \omega_r(0) = 0, \tag{135}$$

*with the following properties:*

*Assume that  $(X, p, \mu_X), (X', p', \mu_{X'}) \in \overline{\mathfrak{MM}}_p$ . Let  $\{z_\alpha\}_{\alpha=1}^{A(\delta)} \subset M$ ,  $\{z'_\alpha\}_{\alpha=1}^{A(\delta)} \subset M'$  be  $\delta$ -nets in  $B(p, r)$ ,  $B'(p', r)$ , correspondingly, and  $\{t_l\}_{l=1}^{L(\delta)}$  be a  $\delta$ -net in  $(\delta, \delta^{-1})$ . Let*

$$|H(z_\alpha, z_\beta, t_l) - H'(z'_\alpha, z'_\beta, t_l)| < \delta, \quad 1 \leq \alpha, \beta \leq A(\delta), \quad 1 \leq l \leq L(\delta), \tag{136}$$

*where  $H, H'$  are the heat kernels on  $X, X'$ , correspondingly.*

*Then*

$$d_{pmGH}((X, p; \mu_X), (X', p'; \mu_{X'})) < \omega_r(\delta), \tag{137}$$

*where  $d_{pmGH}$  is given in Definition 2.8.*



**Proof.** To prove this Theorem it is enough to show that, for any  $\varepsilon > 0$ , there is  $\delta > 0$ , such that if the conditions of the Theorem are satisfied with this  $\delta$ , then (137) is valid with  $\varepsilon$  instead of  $\omega_r(\delta)$ .

Assume that, for some  $\varepsilon > 0$  there are pairs  $(X_i, p_i, \mu_i), (X'_i, p'_i, \mu'_i) \in \overline{\mathfrak{M}\mathfrak{M}}_p$ , which satisfy the conditions of the Theorem with  $i^{-(2s_F+n/2+4)}$  instead of  $\delta$ , but

$$d_{pGH}(X_i, X'_i) \geq 6\varepsilon.$$

Using regular approximations of  $X_i, X'_i$  by pointed manifolds from  $\mathfrak{M}\mathfrak{M}_p$ , there are pairs of pointed manifolds  $(M_i, p_i, \mu_i), (M'_i, p'_i, \mu'_i) \in \mathfrak{M}\mathfrak{M}_p$  which satisfy the conditions of the Theorem with  $1/i$  instead of  $\delta$ , but

$$d_{pmGH}(M_i, M'_i) \geq \varepsilon. \quad (138)$$

Indeed, using Theorem 5.17, we can find  $M_i, M'_i$  such that

$$d_{pmGH}(X_i, M_i), d_{pmGH}(X'_i, M'_i) < \varepsilon/2,$$

so that

$$d_{pmGH}(M_i, M'_i) \geq \varepsilon.$$

Moreover, we can require that there are  $i^{-(2s_F+n/2+4)}$ -nets,  $\{x_\beta^i\} \subset B(p_{X_i}, r), \{y_\beta^i\} \subset B(p_{M_i}, r), \beta = 1, \dots, B$ , and  $\{x_\beta^{i'}\} \subset B(p_{X'_i}, r), \{y_\beta^{i'}\} \subset B(p_{M'_i}, r), \beta = 1, \dots, B'$ , and  $\{t_m\} \subset (i^{-1}, i), m = 1, \dots, M$ , such that

$$\begin{aligned} |H_{M_i}(y_\beta^i, y_{\beta'}^i, t_m) - H_{X_i}(x_\beta^i, x_{\beta'}^i, t_m)| &< i^{-(2s_F+n/2+4)}, \\ |H_{M'_i}(y_\beta^{i'}, y_{\beta'}^{i'}, t_m) - H_{X'_i}(x_\beta^{i'}, x_{\beta'}^{i'}, t_m)| &< i^{-(2s_F+n/2+4)}. \end{aligned} \quad (139)$$

For any  $z_\alpha^i$  involved in the definition of  $d_{pmGH}(X_i, X'_i)$  take a point  $x_{\beta(\alpha)}^i$  which is  $i^{-(2s_F+n/2+4)}$  close to  $z_\alpha^i$  and similar for  $z_\alpha^{i'}$ . Rename  $y_{\beta(\alpha)}^i, y_{\beta(\alpha)}^{i'}$  as  $y_\alpha^i, y_\alpha^{i'}, \alpha = 1, \dots, A(1/i)$ . By the above inequality, for sufficiently large  $i$ , the points  $y_\alpha^i, y_\alpha^{i'}$  form  $2/i$ -net on  $M_i, M'_i$ , correspondingly. Similarly, for any  $t_\ell$  involved in the definition of  $d_{pmGH}(X_i, X'_i)$  take a point  $t_m$  above with  $|t_\ell - t_m| < i^{-(2s_F+n/2+4)}$ . Compare  $H_{M_i}(y_\alpha^i, y_{\alpha'}^i, t_\ell)$  and  $H_{M'_i}(y_\alpha^{i'}, y_{\alpha'}^{i'}, t_\ell)$ ,  $\alpha, \alpha' = 1, \dots, A(1/i)$ . Then, using (136) with  $H_{X_i}, H_{X'_i}$  instead of  $H, H'$ , and (139), we see from (111) that

$$|H_{M_i}(y_\alpha^i, y_{\alpha'}^i, t_\ell) - H_{M'_i}(y_\alpha^{i'}, y_{\alpha'}^{i'}, t_\ell)| < \frac{1}{2i} + C \frac{1}{i^2} < \frac{1}{i},$$

for sufficiently large  $i$ .

Going, if necessary to a subsequence, we can assume that

$$(M_i, p_i; \mu_i) \xrightarrow{f_i} (X, p, \mu_X), \quad (M'_i, p'_i; \mu'_i) \xrightarrow{f'_i} (X', p', \mu_{X'}), \quad (140)$$

in the sense of the pointed measured GH convergence with  $f_i, f'_i$  being the corresponding regular approximations. In particular,

$$B(p_i, r) \xrightarrow{f_i} B(p, r), \quad B'(p'_i, r) \xrightarrow{f'_i} B'(p', r).$$

Let us show that

$$(X, p, ; \mu_X) \simeq (X', p', \mu_{X'}), \quad (141)$$

where  $\simeq$  stands for the measure preserving isometry. This would imply that

$$d_{pmGH}((M_i, p_i, \mu_i), (M'_i, p'_i, \mu'_i)) \rightarrow 0, \quad \text{as } i \rightarrow \infty,$$

contradicting (138).

To prove (141), denote

$$w_\alpha^i = f_i(y_\alpha^i), \quad w_\alpha'^i = f'_i(y_\alpha^i), \quad i = 1, \dots, \alpha = 1, \dots, A(1/i).$$

Let  $\mathbb{P}$  be the set of the double sequences  $p := \{i(k), \alpha(k)\}_{k=1}^\infty, \{i(k)\}_{k=1}^\infty$  being a subsequence of  $\{i\}_{i=1}^\infty$ , such that, with some  $w_p \in B(p, r), w'_p \in B'(p', r)$ , we have

$$w_{\alpha(k)}^{i(k)} \rightarrow w_p, \quad \text{and} \quad w_{\alpha(k)}'^{i(k)} \rightarrow w'_p, \quad \text{as } k \rightarrow \infty. \quad (142)$$

Using estimate (110) in the proof of Theorem 5.17 and Remark 5.18, it follows from (136) that

$$H(w_p, w_{\tilde{p}}, t) = H'(w'_p, w'_{\tilde{p}}, t), \quad p, \tilde{p} \in \mathbb{P}, \quad t > 0. \quad (143)$$

**Lemma 7.2** *Let  $\mathbb{P}$  be the set of convergent double sequences defined above. Then,*

$$\{w_p : p \in \mathbb{P}\} = B(p, r), \quad \{w'_p : p \in \mathbb{P}\} = B'(p', r). \quad (144)$$

**Proof.** Due to (140) and the fact that  $\{z_\alpha^i\}_{\alpha=1}^{A(1/i)}$  form an  $1/i$ -net in  $B(p_i, r)$ , the points  $\{w_\alpha^i\}_{\alpha=1}^{A(1/i)}$  form an  $\delta(i)$ -net in  $B(p, r)$ , where  $\delta(i) \rightarrow 0$  as  $i \rightarrow \infty$  with the similar property valid for  $\{w_\alpha^i\}_{\alpha=1}^{P(i)}$ .

Therefore, for any  $w \in B(p, r)$  there exists a sequence  $\alpha(i)$ ,  $i = 1, \dots$ , such that  $w_{\alpha(i)}^i \rightarrow w$ . Consider the corresponding points  $w_{\alpha(i)}^{i'}$ . Since  $B'(p', r)$  is compact, there is a subsequence  $\{i(k), \alpha(k) = \alpha(i(k))\}_{k=1}^\infty$  such that  $w_{\alpha(i(k))}^{i(k)}$  converge. Then,  $p = \{i(k), \alpha(k)\} \in \mathbb{P}$ , and

$$w_{\alpha(i(k))}^{i(k)} \rightarrow w_p = w, \quad \text{and} \quad w_{\alpha(i(k))}^{i'(k)} \rightarrow w'_p, \quad \text{as } k \rightarrow \infty.$$

This proves the first equation in (144). Changing the role of  $X$  and  $X'$  we obtain the second equation in (144).

QED

**Lemma 7.3** *Let two double sequences,  $p, \hat{p} \in \mathbb{P}$ , satisfy  $w_p = w_{\hat{p}}$ . Then,  $w'_p = w'_{\hat{p}}$ .*

**Proof.** It follows from the conditions of Lemma and (143) that

$$H(w_p, w_{\tilde{p}}, t) = H'(w'_p, w'_{\tilde{p}}, t), \quad H(w_p, w_{\tilde{p}}, t) = H'(w'_{\hat{p}}, w'_{\tilde{p}}, t),$$

for any  $\tilde{p} \in \mathbb{P}$ ,  $t > 0$ . Therefore, by Lemma 7.2,

$$H'(w'_p, \tilde{w}', t) = H'(w'_{\hat{p}}, \tilde{w}', t),$$

for any  $\tilde{w}' \in B'(p', r)$ ,  $t > 0$ . It then follows from Corollary 4.8 that  $w'_p = w'_{\hat{p}}$ .  
QED

Lemma 7.3 makes it possible to introduce an equivalence relation  $\approx$  on  $\mathbb{P}$ ,

$$p \approx \hat{p} \quad \text{iff} \quad w_p = w_{\hat{p}}, \quad w'_p = w'_{\hat{p}}.$$

Then the maps

$$\begin{aligned} \mathcal{F} : \mathbb{P}/\approx &\longrightarrow B(p, r), & \mathcal{F}(p) &= w_p, \\ \mathcal{F}' : \mathbb{P}/\approx &\longrightarrow B'(p', r), & \mathcal{F}'(p) &= w'_p, \end{aligned}$$

are bijections. Thus,  $\Phi = \mathcal{F}' \circ \mathcal{F}^{-1} : B(p, r) \rightarrow B'(p', r)$  is a bijection. Moreover, due to (143),

$$H(w, \hat{w}, t) = H'(\Phi(w), (\Phi(\hat{w}), t), \quad w, \hat{w} \in B(p, r), \quad t > 0. \quad (145)$$

Observe that Theorem 6.2 remains valid if, instead of a dense sequences  $\{w_\alpha\}_{\alpha=1}^\infty$ ,  $\{w'_\alpha\}_{\alpha=1}^\infty$ , we use all points  $w \in B(p, r)$  with the corresponding points  $w'$  running over the whole  $B'(p', r)$ . Thus, using Theorem 6.2,  $\Phi$  can be uniquely extended to an isometry  $\Phi : (X, p) \rightarrow (X', p')$  with

$$\phi_j(x) = \phi'_j(\Phi(x)), \quad j \in \mathbb{Z}_+, \quad \rho(x) = \rho'(\Phi(x)). \quad (146)$$

Theorem 7.1 is proven. QED

Consider next Corollary 1.5. Again, similar to Theorem 1.2, we prove it in a slightly more general form:

**Corollary 7.4** *Let  $(M, p, \mu_M) \in \mathfrak{MM}_p$ . There is  $\delta(M) > 0$  such that, if  $(X, p', \mu_{X'}) \in \overline{\mathfrak{MM}}_p$  satisfies conditions of Theorem 7.1 with  $\delta < \delta(M)$  in (136), then  $X$  is an  $n$ -dimensional manifold  $C_*^3$ -diffeomorphic to  $M$ .*

*Moreover,  $\rho_X = 1$ ,  $h_X \in C_*^2(X)$  and, taking the diffeomorphism between  $M$  and  $X$  to be identity, for any  $\alpha < 1$ ,*

$$\|h_X - h_M\|_{C^{1,\alpha}} \leq \omega^\alpha(\delta). \quad (147)$$

*Here  $\omega^\alpha(\delta)$  is a modulus of continuity function.*

**Proof.** Let  $i_M > 0$  be the injectivity radius of  $M$ . Let  $(X, p_X, \mu_X)$  satisfy (136). Take  $(\widetilde{M}, \widetilde{p}, \widetilde{\mu})$

$$d_{pmGH} \left( (X, p_X, \mu_X), (\widetilde{M}, \widetilde{p}, \widetilde{\mu}) \right) < \delta.$$

Then,

$$d_{pmGH} \left( (\widetilde{M}, \widetilde{p}, \widetilde{\mu}), (M, p, \mu) \right) < 5\delta.$$

Denote by  $f : M \rightarrow \widetilde{M}$ ,  $\widetilde{f} : \widetilde{M} \rightarrow M$  the corresponding approximations.

Let us show that, if  $\delta$  is sufficiently small, then

$$i_{\widetilde{M}} > \frac{1}{3} \min \left( i_M, \pi/3\sqrt{\Lambda} \right). \quad (148)$$

Assume that this equation is not valid, so that there is a closed geodesic,  $\widetilde{\gamma} \subset \widetilde{M}$ , with arclength  $L_{\widetilde{\gamma}} < \frac{4}{5} \min \left( i_M, \pi/3\sqrt{\Lambda} \right)$ . Take  $\widetilde{q}_0 \in \widetilde{\gamma}$  and denote

$\tilde{\xi}_0 \in S_{\tilde{q}_0}^*(\tilde{M})$  its initial direction. Let  $q_0 = \tilde{f}(\tilde{q}_0)$ . Consider the  $\delta$ -dense net  $\{q_k\}_{k=1}^K \subset \partial B_{5i_M/6}(q_0)$ . Take  $\{\tilde{q}_k = f(q_k)\}_{k=1}^K \subset U_{5\delta}(B_{5i_M/6}(\tilde{q}_0))$ , where, for any set  $A \subset \tilde{M}$  and  $a > 0$ ,  $U_a(A)$  is the  $a$ -neighborhood of  $A$ . Consider the set of directions  $\tilde{\xi}_k \in S_{\tilde{q}_0}^*(\tilde{M})$  from  $\tilde{q}_0$  to  $\tilde{q}_k$ ,  $k = 1, \dots, K$ . Then  $\{\tilde{\xi}_k\}$  form  $c(i_M, \Lambda) \delta^{1/n}$ -dense net in  $S_{\tilde{q}_0}^*(\tilde{M})$ , so that, say,

$$d_{S_{\tilde{q}_0}^*(\tilde{M})}(\tilde{\xi}_0, \tilde{\xi}_1) < c\delta^{1/n}.$$

Let  $\tilde{q}' = \tilde{\gamma}(5i_M/6)$ . Then

$$\tilde{d}(\tilde{q}', \tilde{q}_0) \leq \tilde{L}_{\tilde{\gamma}}/2, \quad \tilde{d}(\tilde{q}', \tilde{q}_1) \leq c\delta^{1/n}.$$

On the other hand,

$$\tilde{d}(\tilde{q}', \tilde{q}_0) \geq \tilde{d}(\tilde{q}_1, \tilde{q}_0) - \tilde{d}(\tilde{q}_1, \tilde{q}') \geq 5i_M/6 - 5\delta - c\delta^{1/n} > L_{\tilde{\gamma}},$$

for small  $\delta$ . This proves (148).

This implies that  $X$  is the Gromov-Hausdorff limit of smooth manifolds with bounded below injectivity radii and bounded sectional curvature. By [1], [2],  $X$  is an  $n$ -dimensional manifold with its metric tensor  $h_X$  being  $C_*^2$ -smooth which is, for sufficiently small  $\delta$ ,  $C_*^3$ -diffeomorphic to  $M$ . Moreover, results of [1] then imply all the remaining claims of Corollary.

QED

Consider now the almost isometric maps

$$\Psi^i : (M_i, p_i) \rightarrow (M'_i, p'_i), \quad \Psi_i(x_i) = (f'_i)^{-1} \circ \Phi \circ f_i(x_i), \quad x_i \in M_i, \quad (149)$$

with  $\Psi'_i$  defined analogously. It then follows from (143) together with Theorem 5.17 and Corollary 5.18, that

$$|H_i(x_i, y_i, t) - H'_i(\Psi(x_i), \Psi(y_i), t)| \rightarrow 0, \quad \text{as } i \rightarrow \infty, \quad (150)$$

uniformly with respect to the measured Gromov-Hausdorff distance between  $(M_i, p_i, \mu_i)$  and  $(M'_i, p'_i, \mu'_i)$ . Invoking Theorem 7.1, we see that (150) takes place with respect to the spectral distance  $\delta$  in the conditions of Theorem 7.1.

This makes it possible to prove the following

**Proposition 7.5** *There exists a function  $\tilde{\omega}(\delta)$ , satisfying conditions (135), such that*

$$d'_i(z'^i_\alpha, \Psi_i(z^i_\alpha)) < \tilde{\omega}(\delta), \quad \text{or} \quad d_i(z^i_\alpha, \Psi'_i(z'^i_\alpha)) < \tilde{\omega}(\delta), \quad (151)$$

*if  $\{z^i_\alpha\}, \{z'^i_\alpha\}$  satisfy the conditions of Theorem 7.1.*

**Proof** Assuming the opposite, we find  $\varepsilon > 0$  and sequences  $(M_i, p_i, \mu_i), (M'_i, p'_i, \mu'_i)$  which satisfy the condition of Theorem 7.1 with  $1/i$  instead of  $\delta$  and converge to  $X, X'$ , correspondingly. Similar to the proof of Theorem 7.1,  $X$  and  $X'$  are isometric with the isometry  $\Phi$  uniquely extended from its restriction on  $\{z_\alpha\}$  to the whole  $B(p, r)$ . Then, there is a sequence  $\alpha(i)$  such that

$$d'_i(z'^i_{\alpha(i)}, \Psi_i(z^i_{\alpha(i)})) \geq \varepsilon \quad \text{or} \quad d_i(z^i_{\alpha(i)}, \Psi'_i(z'^i_{\alpha(i)})) \geq \varepsilon. \quad (152)$$

Recall that

$$w^i_{\alpha(i)} = f_i(z^i_{\alpha(i)}), \quad w^h_{\alpha(i)} = f'_i(z'^i_{\alpha(i)}).$$

Then, choosing, as above, a proper double sequence  $p = (i(k), \alpha(k))$ , we obtain that  $\Phi(w_p) = w'_p$ .

By the definition of  $f_i, f'_i$  and  $\Psi_i, \Psi'_i$ , see (150), we have that

$$d'(f'_{i(k)}(\Psi_i(z^{i(k)}_{\alpha(k)})), w'_p) \rightarrow 0, \quad d(f_{i(k)}(\Psi'_i(z'^{i(k)}_{\alpha(k)})), w_p) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This implies that  $d'_i(z'^i_{\alpha(i)}, \Psi_i(z^i_{\alpha(i)})), d_i(z^i_{\alpha(i)}, \Psi'_i(z'^i_{\alpha(i)})) \rightarrow 0$  as  $i \rightarrow \infty$ , contradicting (152).

QED

**Lemma 7.6** *Under the conditions of Theorem 7.1, for any  $\varepsilon > 0$ , there exists  $\delta(\varepsilon) > 0$  such that, for any  $(M, p, \mu), (M', p', \mu') \in \mathfrak{M}$  which satisfy (136) with  $\delta < \delta(\varepsilon)$ ,*

$$\begin{aligned} |H(x, y, t) - H'(\Psi(x), \Psi(y), t)| &< \varepsilon, \\ |H'(x', y', t) - H(\Psi'(x'), \Psi'(y'), t)| &< \varepsilon, \\ \text{for } x, y \in M, \quad x', y' \in M', \quad \varepsilon < t < \varepsilon^{-1}. \end{aligned} \quad (153)$$

**Proof** Assuming the opposite, let  $(M_i, p_i, \mu_i)$ ,  $(M'_i, p'_i, \mu'_i)$  satisfy (136) with  $\delta = 1/i$ , Gromov-Hausdorff converge with measure to  $(X, p, \mu)$ ,  $(X', p', \mu')$  and still do not satisfy (153) with some  $\varepsilon > 0$ . This means that there are, say,  $x_i, y_i \in M_i$  such that

$$|H_i(x_i, y_i, t_i) - H'_i(\Psi_i(x_i), \Psi_i(y_i), t_i)| > \varepsilon.$$

By Theorem 7.1, there is an isometry  $\Phi : X \rightarrow X'$  satisfying (146). By definition (150)

$$f'_i \circ \Psi_i = \Phi \circ f_i. \quad (154)$$

As  $f_i, f'_i$  are almost isometries, it follows from Theorem 5.17, together with (145) that

$$|H_i(x_i, y_i, t) - H'_i(\Psi_i(x_i), \Psi_i(y_i), t)| \rightarrow 0, \quad \text{as } i \rightarrow \infty,$$

uniformly on  $M \times M \times [a, b]$ , for arbitrary  $a, b > 0$ . This contradicts (154).

QED

## Part II

# Orbifold case

## 8 Volume growth and dimensionality of collapse

For given  $n$ , positive numbers  $\Lambda, D, c_0$  and  $1 \leq k \leq n$ , we denote by  $\mathfrak{M}_{k,p} = \mathfrak{M}_{k,p}(n, \Lambda, D; c_0)$  the family of all closed  $n$ -dimensional pointed Riemannian manifolds  $(M, p)$  in  $\mathfrak{M}(n, \Lambda, D)$  such that

$$\mu(B(p, r)) = \frac{\text{Vol}(B(p, r))}{\text{Vol}(M)} \leq c_0 r^{n-k}, \quad (155)$$

for every  $0 < r \leq c_0^{-1}$ .

We start by proving Lemma 1.9.

**Proof of Lemma 1.9.** Let  $\mu_i$  denotes the normalized Riemannian measure of  $M_i$ . We may assume that  $(M_i, p_i, \mu_i)$  converge to  $(X, p, \mu)$  for the pointed measured GH-convergence, where

$$d\mu = \rho \frac{dV}{\text{Vol}(X)}.$$

Suppose first that  $X$  has no singular points. Then there is a uniform positive number  $C$  satisfying

$$\|\rho_X\|_{L^\infty(X)}, \|\rho_X^{-1}\|_{L^\infty(X)} < C,$$

and we have a  $\delta_i$ -almost Riemannian submersion  $f_i : M_i \rightarrow X$  for large  $i$ , i.e,

$$e^{-\delta_i} < \frac{|df_i(\xi)|}{|\xi|} < e^{\delta_i},$$

for all tangent vectors  $\xi$  orthogonal to fibers of  $f_i$ , where  $\lim \delta_i = 0$ , such that  $(f_i)_*(\mu_i) \rightarrow \mu$  for the weak\*-topology. Namely for any open set  $U \subset X$ , we have

$$\mu(U) = \lim_{i \rightarrow \infty} \mu_i(f_i^{-1}(U)). \quad (156)$$

Let  $d = \dim X$ . (156) implies that for any  $0 < r \leq c_0^{-1}$ ,

$$\lim_{i \rightarrow \infty} \mu_i(f_i^{-1}(B(p, r))) = \int_{B(p, r)} d\mu_X \geq C^{-1} \frac{\text{Vol}(B(p, r))}{\text{Vol}(X)} \geq c_1 r^d,$$

where  $c_1$  is a positive number independent of  $p$  and  $r$ . On the other hand, the assumption  $(M_i, p_i) \in \mathfrak{M}_{k,p}(n, \Lambda, D; c_0)$  yields that, for any  $\varepsilon$  and for sufficiently large  $i$ ,

$$\mu_i(f_i^{-1}(B(p, r))) \leq \mu_i(B(p_i, r + \varepsilon)) \leq c_0(r + \varepsilon)^{n-k}.$$

Thus we have, for  $r < c_0$ , that  $c_1 r^d \leq c_0 r^{n-k}$ , and hence  $d \geq (n - k)$ .

For the general case, when  $X$  has singular points, considering the orthonormal frame bundle  $FM_i$  of  $M_i$ , we can proceed as follows. Passing to a subsequence we may assume that the  $O(n)$ -space  $FM_i$  converges to an  $O(n)$ -space  $Y$  with respect to the equivariant GH-topology, where  $Y$  has no singular points. Let  $\tilde{\mu}_i$  be the normalized Riemannian measure of  $FM_i$ . We may



assume that  $(FM_i, \tilde{\mu}_i, O(n))$  converges to  $(Y, \tilde{\mu}, O(n))$  with respect to the equivariant measured GH-topology. Now we have an  $O(n)$ -equivariant  $\delta_i$ -almost Riemannian submersion  $\tilde{f}_i : FM_i \rightarrow Y$  and a map  $f_i : M_i \rightarrow X$  such that  $\pi \circ \tilde{f}_i = f_i \circ \pi_i$ , where  $\pi_i : FM_i \rightarrow M_i$  and  $\pi : Y \rightarrow X$  are the projections along  $O(n)$ -orbits, and  $(\tilde{f}_i)_*(\tilde{\mu}_i) \rightarrow \tilde{\mu}$ .

Take a point  $\tilde{p}_i \in \pi_i^{-1}(p_i)$ . Since  $\pi_i : FM_i \rightarrow M_i$  is a Riemannian submersion with totally geodesic fiber isometric to  $O(n)$ , there exists a constant  $C > 0$  such that

$$\text{Vol}(B(\tilde{p}_i, r)) \leq C \text{Vol}(B(p_i, r)) r^{\dim O(n)}.$$

It follows that for some uniform constant  $C_1$

$$\tilde{\mu}(B(\tilde{p}_i, r)) = \frac{\text{Vol}(B(\tilde{p}_i, r))}{\text{Vol}(FM_i)} \leq \frac{\text{Vol}(B(p_i, r)) r^{\dim O(n)}}{\text{Vol}(O(n)) \text{Vol}(M_i)} \leq C_1 r^{n + \dim O(n) - k},$$

for every  $0 < r \leq c_0^{-1}$ . Applying the previous argument, we obtain  $\dim Y \geq n + \dim O(n) - k$ , and hence  $\dim X = \dim Y - \dim O(n) \geq n - k$  as required. QED

The converse to Lemma 1.9 is also true. Namely we have

**Lemma 8.1** *For given positive integers  $n \geq k$  and  $v_0 > 0$ , there exist positive numbers  $\epsilon_0$  and  $c_0$  satisfying the following: Let  $(n - k)$ -dimensional space  $(X, p) \in \overline{\mathfrak{M}}_p(n, \Lambda, D)$  and  $\text{Vol}(X) \geq v_0 > 0$ . Assume that  $M \in \mathfrak{M}_p(n, \Lambda, D)$  satisfies  $d_{GH}(M, X) < \epsilon_0$ . Then, for any  $q \in M$  and  $0 < r \leq D$ ,*

$$\mu(B(q, r)) \leq c_0 r^{n-k}.$$

*Thus,  $M$  satisfies an extended version of condition (155).*

**Proof.** Suppose the lemma does not hold. Then we have sequences  $M_i$  in  $\mathfrak{M}(n, D)$ ,  $d$ -dimensional spaces  $X_i$  in the closure of  $\mathfrak{M}(n, \Lambda, D)$ ,  $c_i \rightarrow \infty$  and  $0 < r_i \leq D$  such that

$$\text{Vol}(X_i) \geq v_0, \lim_{i \rightarrow \infty} d_{GH}(M_i, X_i) = 0, \mu_i(B(q_i, r_i)) > c_i r_i^d, \quad (157)$$

for some  $q_i \in M_i$ . We may assume that  $X_i$  converges to a space  $X$  in the closure of  $\mathfrak{M}_p(n, \Lambda, D)$ . Note that

$$\dim X = d, \text{Vol}(X) = \lim_{i \rightarrow \infty} \text{Vol}(X_i) \geq v_0, d_{GH}(M_i, X) \rightarrow 0.$$

For large  $i$ , let  $f_i : M_i \rightarrow X$ ,  $\mu_i$ ,  $\mu$  and  $\tilde{f}_i : FM_i \rightarrow Y$ ,  $\tilde{\mu}_i$ ,  $\tilde{\mu}$  be as in the proof of Lemma 1.9. Put  $r := \lim r_i \geq 0$ , and let  $q := \lim f_i(q_i) \in X$ .

First we assume that  $X$  has no singular points, and consider the case of  $r > 0$ . Then, for any  $\varepsilon > 0$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu_i(B(p_i, r_i)) &\leq \lim_{i \rightarrow \infty} \mu_i(B(p_i, r + \varepsilon)) \\ &= \lim_{i \rightarrow \infty} \mu_i(f_i^{-1}(B(q, r + \varepsilon))) = \int_{B(q, r + \varepsilon)} \rho_X \frac{dV_X}{\text{Vol}(X)} \leq C(r + \varepsilon)^d, \end{aligned}$$

for some  $C > 0$ . Then, taking  $\varepsilon \rightarrow 0$ , we see that  $\lim_{i \rightarrow \infty} \mu_i(B(p_i, r_i)) \leq cr^d$ . This is a contradiction to (157).

Next consider the case of  $r = 0$ . Note that there exists a positive number  $C'$  such that

$$\frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(f_i^{-1}(y))} < C', \quad (158)$$

for all  $x, y \in X$  (See Lemma 3.2 in [24]). Since  $f_i$  is a  $\delta_i$ -almost Riemannian submersion with  $\delta_i \rightarrow 0$ , it follows from the co-area formula that

$$\begin{aligned} \text{Vol}(f_i^{-1}(f_i(B(p_i, r_i)))) &\leq 5C \text{Vol}(f_i^{-1}(f_i(p_i))) \text{vol}(f_i(B(p_i, r_i))) \\ &\leq C' \text{Vol}(f_i^{-1}(f_i(p_i))) ((1 + \delta_i) r_i)^d. \end{aligned}$$

Let

$$\rho_i(x) = \frac{\text{Vol}(f_i^{-1}(x))}{\text{Vol}(M_i)}.$$

Since  $\rho_i$  uniformly converges to  $\rho_X$  ([24]), we obtain that, for large  $i$ ,

$$\mu_i(B(q_i, r_i)) \leq \mu_i(f_i^{-1}(f_i(B(q_i, r_i)))) \leq 2C' \rho_X(q) r_i^d.$$

This is a contradiction to (157).

If  $X$  has singular points, we pass to the frame bundle  $FM_i$  and proceed as in the previous argument. First suppose that  $r > 0$ . Then noting  $f_i \circ \pi = \pi \circ \tilde{f}_i$ , we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu_i(B(q_i, r_i)) &= \lim_{i \rightarrow \infty} \mu_i(f_i^{-1}(B(q, r))) \\ &= \lim_{i \rightarrow \infty} \tilde{\mu}_i(F(f_i^{-1}(B(q, r)))) = \tilde{\mu}(\pi^{-1}(B(q, r))) \\ &= \mu(B(q, r)) \leq C_1 \text{Vol}(B(q, r)) \leq C_2 r^d. \end{aligned}$$

This is again a contradiction to (157).

If  $r = 0$ , let

$$\tilde{U}_i = \tilde{f}_i(FB(p_i, r_i)), \quad V_i = \pi_i(\tilde{f}_i^{-1}(\tilde{U}_i)), \quad U_i = f_i(V_i) = f_i(B(p_i, r_i)).$$

Since  $\tilde{f}_i$  is a  $\delta_i$ -almost Riemannian submersion, the co-area formula together with (158) implies that setting  $\hat{q}_i = \tilde{f}_i(\tilde{q}_i)$ ,

$$\text{Vol}(\tilde{f}_i^{-1}(\tilde{U}_i)) \leq C_1 \text{Vol}(\tilde{f}_i^{-1}(\hat{q}_i)) \text{Vol}(\tilde{U}_i).$$

Observe that  $\text{Vol}(\tilde{U}_i) = \text{Vol}(O(n)) \text{Vol}(U_i)$  since  $\pi$ , restricted to  $X^{reg}$ , is a Riemannian submersion with fibers isometric to  $O(n)$ . Put

$$\tilde{\rho}_i(x) := \frac{\text{Vol}(\tilde{f}_i^{-1}(x))}{\text{Vol}(FM_i)}.$$

Since  $\tilde{\rho}_i$  uniformly converges to  $\tilde{\rho}_Y$  ([24]), it follows that

$$\begin{aligned} \mu_i(B(p_i, r_i)) &\leq \mu_i(V_i) = \tilde{\mu}_i(\tilde{f}_i^{-1}(\tilde{U}_i)) \\ &\leq C_3 \tilde{\rho}_i(\hat{q}_i) \text{Vol}(U_i) \leq C_4 ((1 + \delta_i) r_i)^k \leq c_5 r_i^d. \end{aligned}$$

This is a contradiction.

## 9 From isometry to isomorphism

In this section we prove the second part of Theorem 1.7 regarding the isomorphism of isometric orbifolds.

**Theorem 9.1** *Let  $(X, h_X)$  and  $(Y, h_Y)$  be two compact Riemannian orbifolds. Assume that  $X$  and  $Y$  are isometric. Assume also that, either both  $X$  and  $Y$  have no orbifold boundary, or for any  $x \in X$ ,  $y \in Y$  the corresponding groups  $G_x, G_y$  are subgroups of  $SO(n)$ , i.e.  $G_x, G_y$  contain no reflections. Then  $(X, h_X)$  and  $(Y, h_Y)$  are isomorphic.*

We start with a simple observation regarding one-dimensional orbifolds. In this case, one-dimensional compact Riemannian orbifolds are, indeed, compact one-dimensional Riemannian manifolds possibly with boundary, i.e. both  $X$  and  $X'$  are either circles or both are closed interval.

Then, we have the following simple result which we formulate for the convenience of further references:

**Lemma 9.2** *Let  $(X, h)$  and  $(X', h')$  be two isometric one-dimensional compact Riemannian orbifolds with an isometry  $F : X \rightarrow X'$ . Let  $(\tilde{X}, G)$  and  $(\tilde{X}', G')$  be pairs of compact one-dimensional Riemannian manifolds without boundary and finite isometry groups acting on them such that  $\tilde{X}/G = X$ ,  $\tilde{X}'/G' = X'$  and that  $\tilde{X}$  is isometric to  $\tilde{X}'$ . Let  $\tilde{U} \neq \emptyset$ ,  $\tilde{U} \subset \tilde{X}$ , be a connected open set. Assume, in addition, that  $\tilde{f} : \tilde{U} \rightarrow \tilde{X}'$  is an isometric embedding such that*

$$(\pi' \circ \tilde{f})(\tilde{x}) = (F \circ \pi)(\tilde{x}), \quad \text{for } \tilde{x} \in \tilde{U}. \quad (159)$$

*Then, there is a unique Riemannian isometry  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$  which is an extension of  $\tilde{f}$ , i.e.*

$$\tilde{F}|_{\tilde{U}} = \tilde{f},$$

*making the following diagram commutative:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{F}} & \tilde{X}' \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{F} & X'. \end{array} \quad (160)$$

*Here  $\pi : \tilde{X} \rightarrow X$ ,  $\pi' : \tilde{X}' \rightarrow X'$  are the natural projections.*

*If  $\tilde{U} = \emptyset$ , still there is a Riemannian isometry  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$  which satisfies (160), albeit it may be non-unique.*

*In addition,  $G \approx G'$ , where  $\approx$  stands for the group isomorphism.*

**Proof** Suppose first that both  $X$  and  $X'$  are circles of length  $r$ . Then  $\tilde{X}$  and  $\tilde{X}'$  are circles of length  $pr$  for a  $p \in \mathbb{Z}_+$ , and  $G, G'$  are isomorphic cyclic groups generated by rotations. When  $\tilde{U}$  is non-empty, the isometry  $\tilde{f} : \tilde{U} \rightarrow \tilde{X}'$  uniquely extends to an isometry  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$ . It is an easy exercise to check that  $\tilde{F}$  satisfies the desired property. When  $\tilde{U}$  is empty, take  $\tilde{a} \in \tilde{X}$ ,  $\tilde{a}' \in \tilde{X}'$  with  $\pi(\tilde{a}) = \pi'(\tilde{a}')$ , and a neighbourhood  $\tilde{V}$  of  $\tilde{a}$ . It is possible to take an isometric embedding  $\tilde{f} : \tilde{V} \rightarrow \tilde{X}'$  satisfying (160). Therefore this reduces to the previous case.

Suppose next that both  $X$  and  $X'$  are closed interval  $[0, r]$ . Then  $\tilde{X}$  and  $\tilde{X}'$  are circles of length, say  $\ell$ , and  $G, G'$  are dihedral groups generated by rotations and reflections of  $\tilde{X}$  and  $\tilde{X}'$  respectively. Since  $\ell/|G| = \ell/|G'| = r$ , it follows that  $G$  and  $G'$  are isomorphic to each other. It is an easy exercise to extend the previous argument to the present case. QED

## 9.1 Good orbifolds

Recall that  $(X, h)$  is a *good orbifold* in the sense of Thurston, see [63], if there exists a Riemannian manifold  $(\tilde{X}, \tilde{h})$  and a discrete group  $G$  acting on  $\tilde{X}$  by Riemannian automorphisms such that

$$(X, h) = (\tilde{X}, \tilde{h})/G.$$

We denote by  $\pi : \tilde{X} \rightarrow X$  the natural projection from  $\tilde{X}$  to  $\tilde{X}/G$ . So, for example,  $X = \mathbb{S}_r$  is a good orbifold with  $\tilde{X} = \mathbb{R}$  and  $G = \pi r\mathbb{Z}$ .

The main goal of this and the next few subsections is to prove

**Theorem 9.3** *Let  $(X, h) = (\tilde{X}, \tilde{h})/G$  and  $(X', h') = (\tilde{X}', \tilde{h}')/G'$  be two good orbifolds and assume that  $\tilde{X}, \tilde{X}'$  are simply connected. Suppose that either  $X$  and  $X'$  have no orbifold boundary, or  $G, G'$  contain no reflections. Let  $(X, h)$  and  $(X', h')$  be isometric with an isometry  $F : X \rightarrow X'$ . Let  $\tilde{U} \neq \emptyset$ ,  $\tilde{U} \subset \tilde{X}$ , be a connected open set. Assume, in addition, that  $\tilde{f} : \tilde{U} \rightarrow \tilde{X}'$  is an isometric embedding such that*

$$(\pi' \circ \tilde{f})(\tilde{x}) = (F \circ \pi)(\tilde{x}), \quad \text{for } \tilde{x} \in \tilde{U}. \quad (161)$$

*Then there is a unique isometry  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$ , which is an extension of  $\tilde{f}$ , i.e.*

$$\tilde{F}|_{\tilde{U}} = \tilde{f},$$

*making the following diagram commutative:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{F}} & \tilde{X}' \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{F} & X'. \end{array} \quad (162)$$

*If  $\tilde{U} = \emptyset$ , still there is an isometry  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$  which satisfies (162), albeit it may be non-unique.*

*In particular, (162) means that  $G \approx G'$ .*

**Proof.** The proof is by induction in dimension  $n$  of the orbifold and consists of a number of steps and intermediate lemmas. First let us show the theorem for the one-dimensional case. In this case both  $X$  and  $X'$  are either circles of length, say  $\ell$ , or both are closed interval  $[0, \ell]$ . In any case, both  $\tilde{X}$  and  $\tilde{X}'$  are isometric to  $\mathbb{R}$ . In the former case, we may assume that  $G, G'$  are the infinite cyclic group generated by the translation  $x \rightarrow x + \ell$  of  $\mathbb{R}$ . In the later case, we may assume that  $G$  and  $G'$  are the group generated by the translation  $x \rightarrow x + 2\ell$  and the reflection  $x \rightarrow -x$ . In either case, the theorem is certainly valid.

Therefore, let  $n > 1$  and assume that the Theorem is already proven when  $\dim(X), \dim(X') < n$ . In the future, we denote by  $i_0 > 0$  the convex injectivity radii of  $\tilde{X}$  and  $\tilde{X}'$ , so that, for any  $\tilde{x} \in \tilde{X}, \tilde{x}' \in \tilde{X}'$  and  $r < i_0$ , we have  $\tilde{B}(\tilde{x}, r) \subset \tilde{X}, \tilde{B}(\tilde{y}, r) \subset \tilde{X}'$  are convex balls with Riemannian normal coordinates well-defined in these balls. (In the sequel, we will always denote by  $B(p, a)$  a metric ball of radius  $a \geq 0$  centered at a point  $p$  of the corresponding metric space.) Note that although  $\tilde{X}, \tilde{X}'$  may be non-compact, since  $X, X'$  are compact,  $i_0 > 0$ . Indeed, as  $G, G'$  act on  $X, X'$  by isometries, choosing arbitrary  $x_0 \in X, \tilde{x}'_0 \in \tilde{X}'$ , we have

$$i_0 = \min \left( \inf_{\tilde{x} \in \tilde{B}(\tilde{x}_0, 2D)} i(\tilde{x}), \inf_{\tilde{x}' \in \tilde{B}(\tilde{x}'_0, 2D)} i(\tilde{x}') \right) > 0,$$

$$D = \max\{\text{diam}(X), \text{diam}(X')\}.$$

Note that, due to the discreteness of  $G, G'$ , and the fact that  $G, G'$  act on  $X, X'$  by isometries, for any  $\tilde{x} \in \tilde{X}, \tilde{x}' \in \tilde{X}'$ , there are  $r(\tilde{x}) = r(\pi(\tilde{x})), r'(\tilde{x}') = r(\pi'(\tilde{x}'))$  such that, if  $\pi(\tilde{x}_1) = \pi(\tilde{x}_2), \pi'(\tilde{x}'_1) = \pi'(\tilde{x}'_2)$  but  $\tilde{x}_1 \neq \tilde{x}_2, \tilde{x}'_1 \neq \tilde{x}'_2$ , then

$$\begin{aligned} \tilde{B}(\tilde{x}_1, 5r) \cap \tilde{B}(\tilde{x}_2, 5r) &= \emptyset, \text{ if } r < r(x), x = \pi(\tilde{x}_1) = \pi(\tilde{x}_2); \\ \tilde{B}(\tilde{x}'_1, 5r) \cap \tilde{B}(\tilde{x}'_2, 5r) &= \emptyset, \text{ if } r < r'(\tilde{x}'), x' = \pi(\tilde{x}'_1) = \pi(\tilde{x}'_2). \end{aligned} \quad (163)$$

Thus, for any  $x \in X, x' \in X', \tilde{x} \in \pi^{-1}(x), \tilde{x}' \in \pi'^{-1}(x')$  and  $r \leq \min\{r(x), r'(x')\}$ ,

$$B(x, 5r) = \tilde{B}(\tilde{x}, 5r)/G(\tilde{x}), \quad B'(y, 5r) = \tilde{B}'(\tilde{x}', 5r)/G'(\tilde{x}'), \quad (164)$$

where  $G(\tilde{x}), G'(\tilde{x}')$  are the stabiliser subgroups for  $\tilde{x}, \tilde{x}'$ , respectively,

$$G(\tilde{x}) = \{g \in G : g(\tilde{x}) = \tilde{x}\}, \quad G'(\tilde{x}') = \{g' \in G' : g'(\tilde{x}') = \tilde{x}'\}.$$

## 9.2 Ball-to-ball continuation: center point

We start with a special case when  $\tilde{U} = \tilde{B}_0 = \tilde{B}(\tilde{x}_0, r_0)$ , where choose

$$r_0 < \frac{1}{5} \min(r(x_0), r(F(x_0)), i_0), \quad x_0 = \pi(\tilde{x}_0). \quad (165)$$

Then,

$$\begin{aligned} \tilde{f}(\tilde{B}_0) &= \tilde{B}'_0 = \tilde{B}(\tilde{x}'_0, r_0), \quad \tilde{x}'_0 = \tilde{f}(\tilde{x}_0), \\ F(B(x_0, r_0)) &= B(x'_0, r_0), \quad x_0 = \pi(\tilde{x}_0), \quad x'_0 = \pi'(\tilde{x}'_0). \end{aligned}$$

Let now  $\tilde{B}_1 = \tilde{B}(\tilde{x}_1, r_1)$ , where  $r_1$  satisfies (165) for  $x_1 = \pi(\tilde{x}_1)$  and  $x'_1 = F(x_1)$ .

**Lemma 9.4** *Let  $\tilde{B}_0, \tilde{B}_1$  be two balls in  $\tilde{X}$  of radii  $r_0, r_1$  such that*

$$\tilde{B}_0 \cap \tilde{B}_1 \neq \emptyset.$$

*Let  $\tilde{f}|_{\tilde{B}_0}$  satisfy the assumptions of Theorem 9.3. Then, there is a unique Riemannian isometry,  $\tilde{f}_1 : \tilde{U}_1 = \tilde{B}_0 \cup \tilde{B}_1 \rightarrow \tilde{X}'$  such that*

$$\tilde{f}_1|_{\tilde{B}_0} = \tilde{f}, \quad \tilde{f}_1 : \tilde{B}_1 \rightarrow \tilde{B}'_1 \quad (166)$$

$$(\pi' \circ \tilde{f}_1)(\tilde{x}) = (F \circ \pi)(\tilde{x}), \quad \text{for } \tilde{x} \in \tilde{U}_1. \quad (167)$$

Here  $\tilde{B}'_1 = \tilde{B}(\tilde{x}'_1, r_1)$  where

$$\tilde{x}'_1 = \tilde{f}_1(\tilde{x}_1), \quad \pi'(\tilde{x}'_1) = x'_1.$$

We shall prove Lemma 9.4 in the next subsection.

The goal of this subsection is to prove the following

**Proposition 9.5** *Under the conditions of Lemma 9.4, let there exists an isometry  $\tilde{f}_1$  which satisfies (166), (167). Then, the center  $\tilde{x}'_1$  of  $\tilde{B}'_1$  is uniquely determined by  $F$  and  $\tilde{f}$ .*

**Proof** In the following, by a geodesic radius from  $x$  to  $z$ ,  $z \in B(x, r) \subset X$ , we mean a geodesic  $\gamma_{x,\xi}([0, t])$ , such that  $\gamma_{x,\xi}(t) = z$ ,  $t < r$ . We denote such a geodesic radius by  $\gamma_{x,z}$  and use similar notations for  $X', \tilde{X}, \tilde{X}'$ . Observe that, if  $x, z \in X$ ,  $\tilde{x}, \tilde{z} \in \tilde{X}$ , where  $x = \pi(\tilde{x})$ ,  $z = \pi(\tilde{z})$ ,  $d(x, z) = r$  with  $r$  satisfying conditions (165), then

$$\pi \circ (t)\tilde{y}(t) = y(t), \quad 0 \leq t \leq r, \quad \text{where } \tilde{y}(t) = \tilde{\gamma}_{\tilde{x}, \tilde{z}}(t) y(t) = \gamma_{x,z}(t). \quad (168)$$

Moreover, similar relation is valid on  $X', \tilde{X}'$ . Indeed, conditions (165) imply that  $y(t)$  and  $\tilde{y}(t)$  are uniquely determined by

$$d(y(t), x) = t, \quad d(y(t), z) = r - t, \quad \tilde{d}(\tilde{y}(t), \tilde{x}) = t, \quad \tilde{d}(\tilde{y}(t), \tilde{z}) = r - t,$$

which are certainly valid on  $\tilde{\gamma}_{\tilde{x}, \tilde{z}}$ . Since

$$d(y_1, y_2) \leq \tilde{d}(\tilde{y}_1, \tilde{y}_2), \quad y_i = \pi(\tilde{y}_i),$$

the above relations for  $y(t)$  then follow from the triangular inequality, thus implying (168).

Let

$$\tilde{z} \in \tilde{V} = \partial \tilde{B}_1 \cap \tilde{B}_0,$$

and let  $\tilde{w}$  also lies on the geodesic radius  $\tilde{\gamma}_{\tilde{x}_1, \tilde{z}}$ , so that

$$\tilde{w} = \tilde{\gamma}_{\tilde{x}_1, \tilde{z}}(t_1), \quad t_1 < r_1, \quad \tilde{\gamma}_{\tilde{x}_1, \tilde{z}}((t_1, r_1)) \subset \tilde{B}_0,$$

i.e. the geodesic segment between  $\tilde{w}$  and  $\tilde{z}$  is in  $\tilde{B}_0$ .

Let  $B_1 = \pi(\tilde{B}_1)$ ,  $B_0 = \pi(\tilde{B}_0)$ , and  $z = \pi(\tilde{z})$ , etc. Consider next the geodesic  $\tilde{\gamma}_{\tilde{z}, \tilde{w}}$  which coincides with  $\tilde{\gamma}_{\tilde{x}_1, \tilde{z}}$  traversed in the opposite direction, i.e.  $\tilde{\gamma}_{\tilde{z}, \tilde{w}}(r_1) = \tilde{x}_1$ . Note that, by the definition of  $r_1$ , this radius is a minimizing geodesics, with

$$\pi(\tilde{\gamma}_{\tilde{z}, \tilde{w}}(t)) = \gamma_{z,w}(t), \quad 0 \leq t \leq r_1, \quad (169)$$

being the geodesic radius in  $B_1$  i.e. from  $z$  to  $x_1 = \pi(\tilde{x}_1)$ , and

$$\begin{aligned} d(\gamma_{z,w}(s_1), \gamma'_{z,w}(s_2)) &= |s_1 - s_2|, \\ \tilde{d}(\tilde{\gamma}_{\tilde{z}, \tilde{w}}(s_1), \tilde{\gamma}_{\tilde{z}, \tilde{w}}(s_2)) &= |s_1 - s_2|, \quad \text{for } s_1, s_2 \in [0, r_1]. \end{aligned} \quad (170)$$



Let us analyse the corresponding picture on  $X', \tilde{X}'$ . Since  $F$  is an isometry,

$$\gamma'_{z',w'}(t) = F(\gamma_{z,w}(t)), \quad z' = F(z), \quad w' = F(w), \quad 0 \leq t \leq r_1, \quad (171)$$

is a geodesic on  $X'$ . Since also  $\tilde{f}$  is an isometry,

$$\tilde{\gamma}'_{\tilde{z}',\tilde{w}'}(t) = \tilde{f}(\tilde{\gamma}_{\tilde{z},\tilde{w}}(t)), \quad \tilde{z}' = \tilde{f}(\tilde{z}), \quad \tilde{w}' = \tilde{f}(\tilde{w}) \quad 0 \leq t \leq r_1 - t_1, \quad (172)$$

is a geodesic on  $\tilde{X}'$ . Moreover, due to (161),

$$\gamma'_{z',w'}(t) = \pi'(\tilde{\gamma}'_{\tilde{z}',\tilde{w}'}(t)), \quad 0 \leq t \leq r_1 - t_1. \quad (173)$$

In particular,

$$x'_1 = \gamma'_{z',w'}(r_1) = F(x_1). \quad (174)$$

and, as  $F$  is an isometry,  $\gamma'_{z',w'}$  is the geodesic radius  $\gamma'_{x'_1,z'}$ , traversed in the opposite direction.

Our main idea is to continue  $\tilde{\gamma}'_{\tilde{z}',\tilde{w}'}$  beyond  $t = r_1 - t_1$  and to use this continuation to find  $\tilde{x}'_1$ . To this end, consider  $\tilde{\gamma}'_{\tilde{z}',\tilde{w}'}(t)$  for  $0 \leq t \leq r_1$  and define

$$\tilde{x}'_1 = \tilde{\gamma}'_{\tilde{z}',\tilde{w}'}(r_1).$$

We want to show that if an extension  $\tilde{f}$  does exist, then  $\tilde{f}_1(x_1) = \tilde{x}_1$ . To prove that it is necessary to show that

$$\pi'(\tilde{x}'_1) = x'_1. \quad (175)$$

Actually, we intend to show even more, namely, that

$$\pi'(\tilde{\gamma}'_{\tilde{w},\tilde{w}_1}(t)) = \gamma'_{w,w_1}(t), \quad t \in [0, r_1], \quad (176)$$

and to use (176) later in the proof of existence of  $\tilde{f}_1$ . In order to show (176), let

$$t' = \sup\{t \in [0, r_1] : \pi'(\tilde{\gamma}'_{\tilde{w},\tilde{w}_1}(s)) = \gamma'_{w,w_1}(s) \text{ for all } 0 \leq s \leq t\}. \quad (177)$$

Note that, by (173),  $t' \geq t_1 - r_1$ . Our aim is to show that  $t' = r_1$ . Assume, however, that  $t' < r_1$  and let

$$u' = \gamma'_{z',w'}(t'), \quad \tilde{u}' = \tilde{\gamma}'_{\tilde{z}',\tilde{w}'}(t'), \quad u' = \pi'(\tilde{u}'), \quad u' = F(u), \quad \text{with } u = \gamma_{z,w}(t'),$$

where the last equation follows from (171).

Consider the balls  $B'(u', r') \subset X'$ ,  $\tilde{B}'(\tilde{u}', r') \subset \tilde{X}'$ , where  $r'$  is defined by (165) with  $r(u), r(u')$  with, however,  $0 \leq t' - r' < t' + r' \leq r_1$ . Observe that

$$\gamma'_{u', z'}(s) = \gamma'_{z', w'}(t' - s), \quad \gamma'_{u', x'_1}(s) = \gamma'_{z', w'}(t' + s)$$

are geodesic radii of  $B'(u', r')$  and, moreover,

$$d'(\gamma'_{u', z'}(s_1), \gamma'_{u', x'_1}(s_2)) = |s_1 + s_2|, \quad \text{if } s_1, s_2 \in [0, r']. \quad (178)$$

This follows from the corresponding property of  $\gamma_{z, w}(s)$ , see (170), due to (171).

Let now  $\tilde{\mu}(s)$ ,  $0 \leq s \leq r'$ , be a lift of the geodesic radius  $\gamma'_{u', x'_1}(s)$  which starts at  $\tilde{u}'$ ,

$$\pi'(\tilde{\mu}(s)) = \gamma'_{u', x'_1}(s), \quad \tilde{\mu}(0) = \tilde{u}' = \tilde{\gamma}'_{\tilde{z}', \tilde{w}'}(t'). \quad (179)$$

Then,

$$\tilde{d}'(\tilde{\gamma}'_{\tilde{z}', \tilde{w}'}(t' - s_1), \tilde{\mu}(s_2)) \leq s_1 + s_2, \quad s_1, s_2 \in [0, r'], \quad (180)$$

with the equality taking place in (180) if and only if  $\tilde{\mu}(s)$  is the continuation of  $\tilde{\gamma}'_{\tilde{z}', \tilde{w}'}(t)$  beyond  $t = t'$ . However,

$$d'(y'_1, y'_2) \leq \tilde{d}'(\tilde{y}'_1, \tilde{y}'_2), \quad \text{for any } y'_i = \pi'(\tilde{y}'_i), \quad y'_i \in X'.$$

It then follows from (173), (178) and (179) that we have equality in (173). Therefore,

$$\tilde{\mu}(s) = \tilde{\gamma}'_{\tilde{z}', \tilde{w}'}(t' + s), \quad 0 \leq s \leq r'.$$

This shows that  $t' = r_1$  as well as equations (175), (176).

At last, let us show that the result of the above construction is independent of the choice of  $\tilde{z} \in \tilde{B}_0 \cup \partial \tilde{B}_1$ . Assume the opposite and let  $\tilde{z}_i \in \tilde{B}_0 \cup \partial \tilde{B}_1$ ,  $i = 1, 2$ , give rise to different center points  $\tilde{x}'_1 \neq \tilde{x}'_2$  which both satisfy

$$\pi'(\tilde{x}'_1) = x'_1 = \pi'(\tilde{x}'_2), \quad \tilde{d}'(\tilde{x}'_2, \tilde{x}'_1) \geq 5r_1, \quad (181)$$

where the last inequality follows from property (163).

Summarising, we see that  $F, \tilde{f}$  uniquely determine  $\tilde{x}'_1$ .

QED

Note that the above considerations show that all the points  $\tilde{z}' = \tilde{f}(\tilde{z})$ ,  $\tilde{z} \in \tilde{B}_0 \cap \partial\tilde{B}_1$ , lie at the distance  $r_1$  from  $\tilde{x}'_1$ , so that

$$\tilde{f} : \tilde{B}_0 \cap \partial\tilde{B}_1 \mapsto \partial\tilde{B}'_1. \quad (182)$$

### 9.3 Ball-to-ball continuation

In this subsection, using the above construction of  $\tilde{\gamma}'_{\tilde{z}', \tilde{w}'}$ , we will show how to obtain  $\tilde{f}_1$  satisfying (166), (167), thus completing the proof of Lemma 9.4.

We start with

*Case A: Connected  $\partial\tilde{B}_1 \cap \tilde{B}_0$ .*

As  $F$  is an isometry, in particular,

$$\partial f_1 := F|_{\partial B_1} : \partial B_1 \rightarrow \partial B'_1. \quad (183)$$

Moreover, since the inner distance on  $\partial B_1, \partial B'_1$  is uniquely defined by the distances  $d, d'$  on  $X, X'$ , respectively, we see that  $\partial f_1$  is an isometry. In addition,  $\tilde{f}|_{\tilde{V}}$ , where  $\tilde{V} = \partial\tilde{B}_1 \cap \tilde{B}_0$ , satisfies condition (161). Based upon that we obtain

**Proposition 9.6** *Let  $F, \tilde{f}$  satisfy condition of Lemma 9.4 and  $\tilde{V} = \partial\tilde{B}_1 \cap \tilde{B}_0$  is connected. Then, there is a unique isometry*

$$\begin{aligned} \partial\tilde{f}_1 : \partial\tilde{B}_1 &\rightarrow \partial\tilde{B}'_1 : \\ \pi' \circ \partial\tilde{f}_1 &= \partial f_1 \circ \pi, \quad \partial\tilde{f}_1|_{\tilde{V}} = \tilde{f}|_{\tilde{V}}. \end{aligned} \quad (184)$$

**Proof.** As follows from (164),

$$\partial B_1 = \partial\tilde{B}_1/G_{\tilde{x}_1}, \quad \partial B'_1 = \partial\tilde{B}'_1/G'_{\tilde{y}_1}, \quad (185)$$

i.e.  $\partial B_1, \partial B'_1$  are good orbifolds of dimension  $(n-1)$ .

Since, for  $(n-1) \geq 2$ , the sphere  $\partial\tilde{B}_1$  is simply connected, the unique existence of  $\partial\tilde{f}_1$ , satisfying (184), follows from the induction hypothesis.

In the case  $n = 2$ , we observe that  $G(\tilde{x}_1)$ ,  $G'(\tilde{x}'_1)$  are either finite cyclic groups generated by rotations, or else dihedral group generated by rotations and reflections. The lengths  $\tilde{L}$  of the circle  $\partial\tilde{B}_1$  and  $\tilde{L}'$  of the circle  $\partial\tilde{B}'_1$  are given by

$$\tilde{L} = |G|L, \quad \tilde{L}' = |G'|L',$$

where  $L, L'$  are the lengths of  $\partial B_1, \partial B'_1$ . Moreover, as  $\partial f_1$  is an isometry,  $L = L'$ . Therefore, to apply Lemma 9.2, it is enough to show that  $|G| = |G'|$ . To this end we observe that, by the isometry of  $F$ ,

$$V(B(x_1, \rho)) = V'(B'(x'_1, \rho)), \quad \text{for any } \rho \geq 0,$$

where  $V, V'$  denote volumes in  $X, X'$ , respectively. On the other hand,

$$V(B(x_1, \rho)) = \tilde{V}(\tilde{B}(\tilde{x}_1, \rho))/|G|; \quad V'(B'(x'_1, \rho)) = \tilde{V}'(\tilde{B}'(\tilde{x}'_1, \rho))/|G'|,$$

$\tilde{V}, \tilde{V}'$  being volumes in  $\tilde{X}, \tilde{X}'$ . Since

$$\lim_{\rho \rightarrow 0} \frac{\tilde{V}(\tilde{B}(\tilde{x}_1, \rho))}{\rho^n} = \lim_{\rho \rightarrow 0} \frac{\tilde{V}'(\tilde{B}'(\tilde{x}'_1, \rho))}{\rho^n} = c_n,$$

it follows from the preceding considerations that  $|G| = |G'|$ .

QED

Having  $\partial\tilde{f}_1$  at hand, we extend the isometry  $\tilde{f}$  onto  $\tilde{B}_1$ ,

$$\tilde{f}_1 : \tilde{B}_1 \rightarrow \tilde{B}'_1$$

by the process of continuation along radii. Namely, let

$$\tilde{y} = \tilde{y}(t) = \tilde{\gamma}_{\tilde{x}_1, \tilde{z}}(t) \in \tilde{B}_1, \quad 0 \leq t \leq r_1,$$

where  $\tilde{z} \in \partial\tilde{B}_1$ . We define

$$\tilde{y}' = \tilde{y}'(t) := \tilde{f}_1(\tilde{y}(t)) = \tilde{\gamma}'_{\tilde{x}'_1, \tilde{z}'}(t), \quad \tilde{z}' = \partial\tilde{f}_1(\tilde{z}). \quad (186)$$

Note that

$$\tilde{d}(\tilde{y}, \tilde{x}_1) = t, \quad \tilde{d}(\tilde{y}, \tilde{z}) = r_1 - t, \quad (187)$$

and

$$\tilde{d}'(\tilde{y}', \tilde{x}_1') = t, \quad \tilde{d}'(\tilde{y}', \tilde{z}') = r_1 - t. \quad (188)$$

Moreover, since  $r_1 < i_0$ , conditions (187), (188), with  $0 \leq t \leq r_1$ , determine  $\tilde{y}, \tilde{y}'$  uniquely.

Formula (186) defines a bijection

$$\tilde{f}_1 : \tilde{B}_1 \rightarrow \tilde{B}_1',$$

with  $\tilde{f}_1|_{\partial\tilde{B}_1} = \partial\tilde{f}_1$ . Let us show that  $\tilde{f}_1$  satisfies

$$\pi' \circ \tilde{f}_1 = F \circ \pi, \quad \text{in } \tilde{B}_1'. \quad (189)$$

Recall that, since  $\partial f_1$  satisfies (184), we have for  $\tilde{z} \in \partial\tilde{B}_1$  and  $\tilde{z}' \in \partial\tilde{B}_1'$  such that  $\tilde{z}' = \tilde{f}_1(\tilde{z})$ , that

$$z' = F(z), \quad \text{where } z = \pi(\tilde{z}), \quad z' = \pi'(\tilde{z}'). \quad (190)$$

As also  $x_1' = F(x_1)$  and  $F$  is an isometry, this implies that

$$F(\gamma_{x_1, z}(t)) = \gamma'_{x_1', z'}(t), \quad 0 \leq t \leq r_1. \quad (191)$$

Together with (168), this proves (189).

Next we prove that  $\tilde{f}_1$  is a (local) isometry on  $\tilde{B}_1$ . Take  $\tilde{x} \in \tilde{B}_1$  so that

$$x = \pi(\tilde{x}) \in X^{reg}, \quad x' = \pi'(\tilde{x}') \in Y^{reg}, \quad \tilde{x}' = \tilde{f}_1(\tilde{x}), \quad x' = F(x), \quad (192)$$

where the last equations follow from (189). Then there is  $\rho > 0$  such that

$$\pi : \tilde{B}(\tilde{x}, \rho) \rightarrow B(x, \rho); \quad \pi' : \tilde{B}'(\tilde{x}', \rho) \rightarrow B'(x', \rho),$$

are isometries. Using equation (189), this implies that  $\tilde{f}_1 : B(\tilde{x}, \rho) \rightarrow B(\tilde{y}, \rho)$  is an isometry. Therefore,

$$\tilde{h}(\tilde{y}) = \tilde{f}_1^* \left( \tilde{h}'(\tilde{y}') \right), \quad \tilde{y} \in B(\tilde{x}, \rho), \quad \tilde{y}' = \tilde{f}_1(\tilde{y}) \in B(\tilde{x}', \rho). \quad (193)$$

However, since the set of points  $\tilde{x} \in \tilde{B}_1$  which satisfy (192) is dense, condition (193) is satisfied in a dense set of  $\tilde{B}_1$ . At last, due to the smoothness of  $\tilde{h}, \tilde{h}'$ , (193) is valid everywhere on  $\tilde{B}_1$ .

It remains to show that

$$\tilde{f}_1|_{\tilde{B}_0 \cap \tilde{B}_1} = \tilde{f}|_{\tilde{B}_0 \cap \tilde{B}_1}. \quad (194)$$

Recall, see e.g. (182), that every  $\tilde{\gamma}'_{\tilde{z}, \tilde{x}_1}(t)$ ,  $\tilde{z}' = \tilde{f}(\tilde{z})$ ,  $\tilde{z} \in \partial\tilde{B}_1 \cap \tilde{B}_0$ , is the geodesic radius in  $\tilde{B}'_1$  from  $\tilde{z}'$  to  $\tilde{x}_1$ . By (172),

$$\tilde{\gamma}'_{\tilde{z}, \tilde{x}_1}(t) = \tilde{f}(\tilde{\gamma}_{\tilde{z}, \tilde{x}_1}(t)), \quad \text{if } t < t_1(\tilde{z}),$$

if  $t_1(\tilde{z}) > 0$  is defined so that  $\tilde{\gamma}_{\tilde{z}, \tilde{x}_1}(t) \in \tilde{B}_0$  for  $0 \leq t < t_1$ . On the other hand, by the definition of  $\tilde{f}_1$ , see (187), (188),

$$\tilde{\gamma}'_{\tilde{w}, \tilde{y}_1}(t) = \tilde{f}_1(\tilde{\gamma}_{\tilde{z}, \tilde{x}_1}(t)), \quad 0 \leq t \leq r_1.$$

Comparing these two equalities, we see that

$$\tilde{f}_1(\tilde{\gamma}_{\tilde{z}, \tilde{x}_1}(t)) = \tilde{f}(\tilde{\gamma}_{\tilde{z}, \tilde{x}_1}(t)), \quad 0 \leq t < t_1(\tilde{z}).$$

Therefore, there is an open ball  $\tilde{B}(\tilde{y}, \rho)$ ,  $0 < \rho < \frac{1}{5} \min(r(y), r(F(y)), i_0)$  such that

$$\tilde{B}(\tilde{y}, \rho) \subset \tilde{B}_1 \cap \tilde{B}_0, \quad \tilde{f}_1|_{\tilde{B}(\tilde{x}, \rho)} = \tilde{f}|_{\tilde{B}(\tilde{x}, \rho)}. \quad (195)$$

Since  $\tilde{B}_1 \cap \tilde{B}_0$  is convex, we can use Proposition 3.62, [51] on the properties of isometries which implies equation (194).

This completes the proof of Lemma 9.4 in *Case A*.

*Case B: General.* Assume now that  $\tilde{B}_0 \cap \partial\tilde{B}_1$  is not connected. Let  $\tilde{V} \subset \tilde{B}_0 \cap \partial\tilde{B}_1$  be open and connected. Similar to *Case A*, let  $\partial\tilde{f}_{\tilde{V}} : \partial\tilde{B}_1 \rightarrow \tilde{B}'_1$  be an isometry which satisfies (184) and

$$\partial\tilde{f}_{\tilde{V}}|_{\tilde{V}} = \tilde{f}|_{\tilde{V}}.$$

By the same construction as in *Case A* we extend it to an isomorphism  $\tilde{f}_{\tilde{V}} : \tilde{B}_1 \rightarrow \tilde{B}'_1$  and show that

$$\tilde{f}|_{\tilde{V}} = \tilde{f}_{\tilde{V}}|_{\tilde{V}}. \quad (196)$$

It remains to show that this extension does not depend on  $\tilde{V}$ . Indeed, if  $\tilde{V}' \subset \tilde{B}_0 \cap \partial\tilde{B}_1$  be another subset of  $\partial\tilde{B}_1$ , then we can construct another extension  $\tilde{f}_{\tilde{V}'}$  of  $\tilde{f}$  which also satisfies (196). But then

$$\tilde{f}_{\tilde{V}} = \tilde{f}_{\tilde{V}'} \quad \text{on } \tilde{B}_0 \cap \tilde{B}_1.$$

Since  $\tilde{B}_0 \cup \tilde{B}_1$  is connected, Proposition 3.62, [51] implies that  $\tilde{f}_{\tilde{V}} = \tilde{f}_{\tilde{V}}$  everywhere on  $\tilde{B}_0 \cup \tilde{B}_1$ .

Defining  $\tilde{f}_1 = \tilde{f}_{\tilde{V}}$  for any  $\tilde{V}$ , we complete the proof of Lemma 9.4. QED

## 9.4 Continuation along balls.

Consider now a general open connected  $\tilde{U} \neq \emptyset$  with  $\tilde{f}$  satisfying (161). Our next foal is to extend  $\tilde{f}$  onto the whole  $\tilde{X}$ . To this end, we first choose an arbitrary ball  $\tilde{B}_0 = \tilde{B}(\tilde{x}_0, r_0)$ ,

$$r_0 < \frac{1}{5} \min(r(x_0), r(F(x_0)), i_0), \quad (197)$$

where  $\tilde{x}_0 = \pi(\tilde{x}_0)$  such that  $\tilde{B}_0 \subset \tilde{U}$ .

Next, let  $\tilde{x} \in \tilde{X}$  be arbitrary and let  $\tilde{\mu}(t)$ ,  $\tilde{\mu}(0) = \tilde{x}_0$ ,  $\tilde{\mu}(1) = \tilde{x}$ , be a connecting curve.

**Definition 9.7** *A (finite) chain of balls  $\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_p$  is called associated with  $\tilde{\mu}$  if there are intervals  $[0, t_0^+)$ ,  $(t_1^-, t_1^+)$ ,  $\dots$ ,  $(t_p^-, 1]$  with  $t_j^+ \in (t_{(j+1)}^-, t_{(j+1)}^+)$ , such that*

$$\tilde{\mu}(t) \in \tilde{B}_j, \quad t \in (t_j^-, t_j^+), \quad j = 0, \dots, p.$$

Here assume also that each ball  $\tilde{B}_j$  satisfies condition (197).

The idea of the extension of  $\tilde{f}$  to (a vicinity of)  $\tilde{x}$  is to extend it, using the construction described in subsection 9.3, along a chain of balls associated with  $\tilde{\mu}$ . Observe that then

$$\tilde{f}_j|_{\tilde{B}_j \cap \tilde{B}_{j+1}} = \tilde{f}_{j+1}|_{\tilde{B}_j \cap \tilde{B}_{j+1}},$$

where  $\tilde{f}_j$  is the continuation  $\tilde{f}_0 = \tilde{f}$  to the ball  $\tilde{B}_j$ . Note that we can always assume  $t_j^+ \leq t_{(j+2)}^-$  since otherwise the piece  $\tilde{\mu}(\min(t_{(j+1)}^-, t_{(j+2)}^-), t_{(j)}^+) \subset (\tilde{B}_j \cap \tilde{B}_{(j+1)} \cap \tilde{B}_{(j+2)})$ , and the above equality implies that

$$\begin{aligned} \tilde{f}_j|_{\tilde{B}_j \cap \tilde{B}_{j+1} \cap \tilde{B}_{j+2}} &= \tilde{f}_{j+1}|_{\tilde{B}_j \cap \tilde{B}_{j+1} \cap \tilde{B}_{j+2}}, \\ \tilde{f}_{j+1}|_{\tilde{B}_j \cap \tilde{B}_{j+1} \cap \tilde{B}_{j+2}} &= \tilde{f}_{j+2}|_{\tilde{B}_j \cap \tilde{B}_{j+1} \cap \tilde{B}_{j+2}}. \end{aligned}$$

Thus, by Proposition 3.62, [51],

$$\tilde{f}_j|_{\tilde{B}_j \cap \tilde{B}_{j+2}} = \tilde{f}_{j+2}|_{\tilde{B}_j \cap \tilde{B}_{j+2}}$$

and, with omitted  $\tilde{B}_{(j+1)}$ , we obtain the same continuation to the vicinity of  $\tilde{x}$ .

The above observations show that, for any  $0 \leq t \leq 1$ , we construct an unique local isometry  $\tilde{f}^t : \tilde{U}^t \rightarrow \tilde{X}'$ , where  $\tilde{U}^t$  is an open convex vicinity of  $\mu(t)$ . This  $\tilde{f}^t$  is a continuation of  $\tilde{f}$  along a chosen set of balls and, if  $t_j^- < t < t_j^+$ , then  $\tilde{f}^t$  coincides with  $\tilde{f}_j$  near  $\mu(t)$ . However, at this stage, we do not assume that, if  $\tilde{\mu}(t) = \tilde{\mu}(t')$ , then  $\tilde{f}^t$  coincides with  $\tilde{f}^{t'}$  near  $\tilde{\mu}(t) = \tilde{\mu}(t')$ .

Let us first show an existence of an associated chain of balls for any finite curve  $\tilde{\mu}$ . Let  $t_0 = \sup t \geq 0$  such that there is a finite chain of balls associated with the curve  $\tilde{\mu}[0, t]$ . We need to show that  $t_0 = 1$ . Assume the opposite and consider a ball  $\tilde{B}' := B(\tilde{x}', r') \subset \tilde{X}$ , where  $\tilde{x}' = \tilde{\mu}(t_0)$  and  $r'$  satisfies (197). Then, there are  $t_- < t_0 < t_+$ , such that

$$\tilde{\mu}(t) \in \tilde{B}', \quad \text{if } t_- < t < t_+.$$

On the other hand, for any  $\tilde{t}$ ,  $t_- < \tilde{t} < t_0$ , there is a chain of balls  $\tilde{B}_0, \dots, \tilde{B}_p$  associated with  $\tilde{\mu}[0, \tilde{t}]$ . Then the chain of balls  $\tilde{B}_0, \dots, \tilde{B}_p, \tilde{B}_{p+1} = \tilde{B}'$  is associated with  $\tilde{\mu}[0, s]$ , for any  $t_0 < s < t_+$ , which contradicts the definition of  $t_0$ .

Therefore, by the above procedure, it is possible to extend  $\tilde{f}$  to a vicinity of any point  $\tilde{x} \in \tilde{X}$  providing a local isometry,  $\tilde{f}_{\tilde{x}}$  in this vicinity, i.e. there is  $\tilde{U}(\tilde{x}) \subset \tilde{X}$ ,  $\tilde{x} \in \tilde{U}(\tilde{x})$  with

$$\tilde{f}_{\tilde{x}} : \tilde{U}(\tilde{x}) \rightarrow \tilde{X}', \quad F \circ \pi|_{\tilde{U}(\tilde{x})} = \pi' \circ \tilde{f}_{\tilde{x}}|_{\tilde{U}(\tilde{x})}.$$

Note that, at this stage, we do not assume that, for  $\tilde{x} \in \tilde{X}$ , such a local isometry is unique. To show this we start with the following result:

**Lemma 9.8** *Let, for a given curve  $\tilde{\mu}[0, 1]$  connecting  $\tilde{x}_0$  and  $\tilde{x}$ ,  $\{\tilde{B}_i\}_{i=0}^p, \{\tilde{D}_j\}_{j=0}^q$  be associated chains of balls in the sense of definition 9.7. Denote by  $\tilde{f}_B^t, \tilde{f}_D^t$  the corresponding extensions of  $\tilde{f}$  to a vicinity of  $\tilde{\mu}(t)$ .*

*Then, for all  $t \in [0, 1]$ ,*

$$\tilde{f}_B^t = \tilde{f}_D^t \quad \text{on } \tilde{U}_B^t \cap \tilde{U}_D^t.$$



**Proof.** Let

$$t^* = \sup\{t \in [0, 1] : \tilde{f}_B^s = \tilde{f}_D^s, \text{ near } \tilde{\mu}(s), \text{ for } 0 \leq s \leq t\}.$$

Clearly,  $t^* > 0$  since, for small  $t'$ ,  $\tilde{f}_B^t, \tilde{f}_D^t$  coincide with  $\tilde{f}$ . Let  $t^* < 1$  and  $\tilde{B}^*, \tilde{D}^*$  be the balls in the associated chains which correspond to  $t^*$ . We note that, in principle, there may be two balls in every chain which correspond to  $t^*$ , in which case we choose any of them. Denote by  $\tilde{f}_B^*, \tilde{f}_D^*$  the continuations of  $\tilde{f}$  onto balls  $\tilde{B}^*, \tilde{D}^*$  along the chains  $\{\tilde{B}_i\}_{i=0}^p, \{\tilde{D}_j\}_{j=0}^q$ . Note that there is  $\varepsilon > 0$  such that

$$\tilde{\mu}(t^* - \varepsilon, t^* + \varepsilon) \subset \tilde{B}^* \cap \tilde{D}^*.$$

Choose any  $t \in (t^* - \varepsilon, t^*)$ . Then  $\tilde{f}_B^t = \tilde{f}_D^t$  in some small ball  $\tilde{B}(\tilde{\mu}(t), \delta) \subset \tilde{B}^* \cap \tilde{D}^*$ . However, according to our construction of  $\tilde{f}_B^t, \tilde{f}_D^t$ , then

$$\tilde{f}_B^*|_{\tilde{B}(\tilde{\mu}(t), \delta)} = \tilde{f}_B^t|_{\tilde{B}(\tilde{\mu}(t), \delta)} = \tilde{f}_D^t|_{\tilde{B}(\tilde{\mu}(t), \delta)} = \tilde{f}_D^*|_{\tilde{B}(\tilde{\mu}(t), \delta)}.$$

As  $\tilde{B}^* \cap \tilde{D}^*$  is convex, by Proposition 3.62, [51],

$$\tilde{f}_B|_{\tilde{B}^* \cap \tilde{D}^*} = \tilde{f}_D|_{\tilde{B}^* \cap \tilde{D}^*},$$

and, in particular, near  $\tilde{\mu}(t), t^* < t < t^* + \varepsilon$ . As this contradicts the definition of  $t^*$ . Therefore,  $t^* = 1$ . QED

Note that, as follows from the proof of Lemma 9.8, for any  $t \in [0, 1]$  there is  $\rho(t) > 0$ , such that, for choice of the chain of balls  $\{\tilde{B}_i\}_{i=0}^p$ ,  $\tilde{f}_B^t$  can be uniquely extended as a Riemannian isometry, satisfying (162), onto the convex ball  $\tilde{B}(\mu(t), \rho(t))$ . Using these results, we can show

**Lemma 9.9** *Let  $F, \tilde{f}$  and  $\tilde{U} \neq \emptyset$  satisfy the conditions of Theorem 9.3. Then, for any  $\tilde{x} \in \tilde{X}$ , the local isometric extension  $\tilde{f}_{\tilde{x}}$  is uniquely defined. Let*

$$\tilde{F}(\tilde{x}) = \tilde{f}_{\tilde{x}}(\tilde{x}), \quad \tilde{x} \in \tilde{X}. \tag{198}$$

*Then  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$  is a local isometry satisfying (162) and*

$$\tilde{F}|_{\tilde{U}} = \tilde{f}. \tag{199}$$

**Proof** Let us first show that  $\tilde{f}_{\tilde{x}}$  does not depend upon the choice of a curve  $\tilde{\mu}$  connecting  $\tilde{x}_0$  to  $\tilde{x}$ . Assume that  $\tilde{\mu}_0(t)$  and  $\tilde{\mu}_1(t)$  are two curves from  $\tilde{x}_0$  to  $\tilde{x}$  and denote by  $\tilde{f}_{\tilde{x}}^0$  and  $\tilde{f}_{\tilde{x}}^1$  the corresponding continuations of  $\tilde{f}$  along  $\tilde{\mu}_0(t)$  and  $\tilde{\mu}_1(t)$  which, by Lemma 9.8, do not depend upon a choice of chains associated with these curves. We want to show that there is  $\rho = \rho(\tilde{x}) > 0$  so that  $\tilde{f}_{\tilde{x}}^0, \tilde{f}_{\tilde{x}}^1$  have a unique continuation to  $\tilde{B}(\tilde{x}, \rho)$  and

$$\tilde{f}_{\tilde{x}}^0|_{\tilde{B}(\tilde{x}, \rho)} = \tilde{f}_{\tilde{x}}^1|_{\tilde{B}(\tilde{x}, \rho)}. \quad (200)$$

Since  $\tilde{X}$  is simply connected, we can deform  $\tilde{\mu}_0$  to  $\tilde{\mu}_1$  as  $\tilde{\mu}_s(t)$ ,  $0 \leq s \leq 1$ . Observe now that if  $\{\tilde{B}_j(s)\}$  is a chain associated with a curve  $\tilde{\mu}_s$ , then it is associated also with  $\tilde{\mu}_\sigma$ , for  $\sigma$  sufficiently close to  $s$ . Therefore, for any  $s \in [0, 1]$ , there is an (relatively) open interval  $I_s \subset [0, 1]$ ,  $s \in I_s$  such that  $\tilde{f}^\sigma = \tilde{f}^s$ , for  $\sigma \in I_s$ . Here we denote by  $\tilde{f}^s$  the local isometry  $\tilde{f}_{\tilde{x}}$  obtained by continuation along  $\tilde{\mu}_\sigma$ . Using the connectedness and compactness of the interval  $s \in [0, 1]$ , we see that  $\tilde{f}^0 = \tilde{f}^s$  near  $\tilde{x}$ . Moreover, since  $\tilde{f}^0 = \tilde{f}^s$  near  $\tilde{x}$  implies that their unique continuations coincide on e.g.  $\tilde{B}(\tilde{x}, \rho)$ , where  $\rho$  is independent of  $s$ , we obtain (200) proving that  $\tilde{f}_{\tilde{x}}$  is defined uniquely.

Let now  $\tilde{F}$  be of form (198). Then using the fact that, for  $\tilde{x} \in \tilde{X}$  and a chain  $\{\tilde{B}_i\}_{i=0}^p$  from  $\tilde{x}_0$  to  $\tilde{x}$ , the same chain is appropriate for  $\tilde{y}$  near  $\tilde{x}$ , by the similar arguments as above we see that

$$\tilde{F} = \tilde{f}_{\tilde{x}} \quad \text{in a vicinity of } \tilde{x}.$$

This shows that  $\tilde{F}$  is a local isometry satisfying (162).

At last, we use the fact that  $\tilde{U}$  is connected so that we can choose  $\tilde{\mu}[0, 1] \subset \tilde{U}$ . In this case, we can choose  $\tilde{B}_j \subset \tilde{U}$ ,  $j = 0, \dots, p$ . Then,  $\tilde{f}_B^t$  coincide with  $\tilde{f}$  for any  $t \in [0, 1]$ . This proves (199).

QED

We complete this subsection by considering the case  $\tilde{U} = \emptyset$ . Take  $x_0 \in X^{reg}$ ,  $x'_0 = F(x_0) \in X'^{reg}$ . Take  $\tilde{x}_0 \in (\pi)^{-1}(x_0)$ ,  $\tilde{x}'_0 \in (\pi')^{-1}(x'_0)$  and let  $r_0 > 0$  satisfying (197) be so small that

$$\pi : \tilde{B}(\tilde{x}_0, r_0) \rightarrow B(x_0, r_0), \quad \pi' : \tilde{B}(\tilde{x}'_0, r_0) \rightarrow B(x'_0, r_0),$$

are isometric coverings. Since  $F$  is an isometry, we can take

$$\tilde{f} = (\pi')^{-1}|_{\tilde{B}(\tilde{x}'_0, r_0)} \circ F \circ \pi|_{\tilde{B}(\tilde{x}_0, r_0)}, \quad \tilde{f} : \tilde{B}(\tilde{x}_0, r_0) \rightarrow \tilde{X}',$$

to be the desired local isometry with  $\tilde{U} = \tilde{B}(\tilde{x}_0, r_0)$ . Since, by definition,  $\tilde{f}$  satisfies (161), by using the constructions of subsections 9.2–9.4, we construct  $\tilde{F}$  which satisfies (162).

## 9.5 Completing proof of Theorem 9.3

It remains to show that the constructed  $\tilde{F} : \tilde{X} \rightarrow \tilde{Y}$  is an isometry rather than a local isometry.

Let us first show that  $\tilde{F}(\tilde{X}) = \tilde{X}'$ , i.e.  $\tilde{F}$  is surjective. Since  $\tilde{X}'$  is connected and, by local isometry,  $\tilde{F}(\tilde{X})$  is open, it is enough to show that  $\tilde{F}(\tilde{X})$  is closed.

Let  $\tilde{y}' \in \text{cl}(\tilde{F}(\tilde{X})) \setminus \tilde{F}(\tilde{X})$  with  $y' = \pi'(\tilde{y})$ ,  $y = F^{-1}(y')$ . Choose  $r > 0$  satisfying conditions (165). Let now  $\tilde{y}'_0 \in \tilde{F}(\tilde{X}) \cap \tilde{B}(\tilde{y}', r)$  and  $\tilde{y}_0 \in (\tilde{F})^{-1}(\tilde{y}'_0)$ . Consider  $y = \pi(\tilde{y})$ . Then, since  $X = \tilde{X}/G$ , it follows from conditions (165) that there is a unique  $\tilde{y} \in \tilde{X}$  such that

$$\tilde{d}(\tilde{y}, \tilde{y}_0) < r. \quad (201)$$

Then

$$\tilde{F}(\tilde{y}) = \tilde{y}'. \quad (202)$$

Indeed, since  $\tilde{F}$  is a local isometry,

$$\tilde{d}(\tilde{F}(\tilde{y}), \tilde{y}'_0) < r, \quad (203)$$

where we use that  $\tilde{F}(\tilde{y}_0) = \tilde{y}'_0$ . Since  $\pi'(\tilde{F}(\tilde{y})) = \pi'(\tilde{y}')$ , (202) follows from (201), (203) due to (165).

Summarising,  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$  is a surjective local isometry. Thus,  $\tilde{F}$  is a Riemannian covering of  $\tilde{X}'$ . However,  $\tilde{X}'$  is simply connected, therefore,  $\tilde{F}$  is a Riemannian isometry between  $\tilde{X}$  and  $\tilde{X}'$ .

It remains to show that

$$G \approx G'. \quad (204)$$

Indeed, for any  $g \in G$ , let

$$g' = \tilde{F} \circ g \circ (\tilde{F})^{-1}.$$

Note that  $g'$  is an isometry in  $\tilde{X}'$ . To show that  $g' \in G'$  we should prove  $\pi' \circ g' = \pi'$ . We have

$$\begin{aligned}\pi' \circ g' &= \pi' \circ \tilde{F} \circ g \circ (\tilde{F})^{-1} = F \circ \pi \circ g \circ (\tilde{F})^{-1} \\ &= F \circ \pi \circ (\tilde{F})^{-1} = \pi' \circ \tilde{F} \circ (\tilde{F})^{-1} = \pi'.\end{aligned}$$

In proving the above we have twice made use of equation (162), which is already at hand, and also the fact that, since  $X = \tilde{X}/G$ , we have  $\pi \circ g = \pi$ . Clearly,

$$g'_1 \circ g'_2 = \left( \tilde{F} \circ g_1 \circ (\tilde{F})^{-1} \right) \circ \left( \tilde{F} \circ g_2 \circ (\tilde{F})^{-1} \right) = \tilde{F} \circ g_1 \circ g_2 \circ (\tilde{F})^{-1}.$$

Thus, the map  $g \rightarrow \tilde{F} \circ g \circ (\tilde{F})^{-1}$  is a homomorphism from  $G$  to  $G'$  with the inverse  $g' \rightarrow (\tilde{F})^{-1} \circ g' \circ \tilde{F}$ . This proves (204).

This completes the proof of Theorem 9.3.

QED

## 9.6 General Case

To complete the proof of the theorem, consider now two, not necessarily good, orbifolds  $X$  and  $X'$  which are isometric with an isometry  $F$ . For any  $x \in X$  denote by  $x' \in X'$  the corresponding point  $x' = F(x)$ . Choose  $r > 0$  sufficiently small so that the balls  $V = B(x, 2r)$ ,  $V' = B'(x', 2r)$  have uniformizing covers

$$\tilde{V} = \tilde{B}(O, 2r), \quad \tilde{V}' = \tilde{B}'(O', 2r), \quad (205)$$

where  $O, O'$  are just two copies of  $O$ . Consider the balls  $U = B(x, r)$ ,  $U' = B'(x', r)$ . We can treat  $U, U'$  as two good orbifolds,

$$U = \tilde{B}(O, 2r)/G, \quad U' = \tilde{B}'(O', 2r)/G', \quad G = G(x), \quad G' = G(x').$$

Then  $F_U := F|_U$  is an isometry between them  $U$  and  $U'$ . We intend to apply Theorem 9.3 to this case. However, since  $U, U'$  are not complete we should make the necessary modifications.

Namely, the existence of a uniform  $i_0$  should be substituted by  $i_0(d_\partial)$ , where  $d_\partial$  is the distance to  $\partial U, \partial U'$ . This function  $i_0(d_\partial)$  is strictly positive inside  $U, U'$ . Using  $i_0(d_\partial)$  we correspondingly modify equation (165).

With this modification is it possible to carry out all the considerations of Theorem 9.3 for  $U, U'$ , so that  $F_U$  can be lifted to an isometry

$$\tilde{F}_U : \tilde{U} \rightarrow \tilde{U}', \quad F_U \circ \pi = \pi' \circ \tilde{F}_U,$$

with

$$(\tilde{F}_U)_* : G \rightarrow G', \quad (\tilde{F}_U)_*(g) = \tilde{F}_U \circ g \circ (\tilde{F}_U)^{-1},$$

being an isomorphism between  $G = G(x)$  and  $G' = G(x')$ .

As this is true for any  $x \in X$ , this completes the proof of the second part of Theorem 1.7.

QED

Finally we give a proof of Corollary 1.10.

**Proof of Corollary 1.10** Suppose that the corollary does not hold. Then we would have sequences  $(M_i, p_i, \mu_{M_i})$  and  $(M'_i, p'_i, \mu_{M'_i})$  in  $\mathfrak{MM}_p \cap \mathfrak{M}_{n-1}(n, \Lambda, D; c_0)$  satisfying

$$d_{pmGH}((M_i, p_i, \mu_{M_i}), (M'_i, p'_i, \mu_{M'_i})) < 1/i$$

which, however, do not satisfy the conclusion of the corollary. Then, by the precompactness of  $\mathfrak{MM}_p$ , passing to subsequences we may assume that  $(M_i, p_i, \mu_{M_i})$  and  $(M'_i, p'_i, \mu_{M'_i})$  converge to  $(X, p, \mu)$  and  $(X', p', \mu')$  respectively with respect to the pointed measured GH-distance. Since

$$d_{pmGH}((X, p, \mu), (X', p', \mu')) = 0,$$

from Lemma 2.11 we have an isometry  $\Phi : (X, p) \rightarrow (X', p')$  such that  $\Phi_*(\mu) = \mu'$ . By Lemma 1.9,  $X$  has dimension  $\geq n - 1$ . If  $\dim X = n$ ,  $X$  and  $X'$  are manifold, and  $M_i$  and  $M'_i$  are diffeomorphic to each other for large  $i$ . Suppose  $\dim X = n - 1$ . By Corollaries 2.4, 2.7,  $X$  and  $X'$  are orbifolds and  $M_i$  and  $M'_i$  are Seifert  $S^1$ -bundle over  $X$  and  $X'$  respectively. This is a contradiction.

QED

## 10 Appendix A: Collapsing manifolds in physics

In the modern quantum field theory, in particular in the string theory one often models the Universe as a high-dimensional, almost collapsed manifold.

This type of considerations started from the Kaluza-Klein theory in 1921 in which the 5-dimensional Einstein equations are considered on  $\mathbb{R}^4 \times S^1(\varepsilon)$ , that is, on the cartesian product of the standard 4-dimensional space-time with the Minkowski metric and a circle  $S^1(\varepsilon)$  of radius  $\varepsilon$ . As  $\varepsilon \rightarrow 0$ , the 5-dimensional Einstein equation yields to a model containing both the Einstein equation and Maxwell's equations. In this appendix shortly review this and discuss its relation to the considerations presented in the main text of this paper.

In the Kaluza-Klein theory one starts with a 5-dimensional manifold  $N = \mathbb{R}^4 \times S^1$  with the metric  $\hat{g} = \hat{g}_{jk}$ ,  $j, k = 0, 1, 2, 3, 4$  which is a Lorentzian metric of the type  $(-, +, +, +, +)$ . The "hat" marks the fact that  $\hat{g}$  is defined on a five dimensional manifold. We use on  $N$  the coordinates  $x = (y^0, y^1, y^2, y^3, \theta)$ , where  $\theta$  is consider as a variable having values on  $[0, 2\pi]$ . Let us start with a background metric (or the non-perturbed metric)

$$\hat{g}_{jk}(x)dx^jdx^k = \eta_{\nu\mu}(y, \theta)dy^\nu dy^\mu + \varepsilon^2 d\theta^2,$$

where  $\varepsilon > 0$  a small constant and  $\nu, \mu, \alpha, \beta$  are summing indexes having values  $\nu, \mu, \alpha, \beta \in \{0, 1, 2, 3\}$ . Here, we consider the case when  $[\eta_{\nu\mu}(y, \theta)]_{\nu, \mu=0}^3 = \text{diag}(-1, 1, 1, 1)$ . Next we consider perturbations of the background metric  $\hat{g}$  that we write in the form

$$\hat{g}_{jk}(x)dx^jdx^k = e^{-\sigma/3}(e^\sigma(d\theta + \kappa A_\mu dy^\mu)^2 + g_{\nu\mu}dy^\nu dy^\mu),$$

where  $\kappa > 0$  is a constant,  $\sigma = \sigma(y, \theta)$  is a function close to the constant  $c(\varepsilon) := 3 \log \varepsilon$ ,  $\varepsilon > 0$  the 1-form  $A = A_\mu(y, \theta)dy^\mu$  is small and  $g_{\nu\mu}(y, \theta)$  is close to  $e^{\sigma(y, \theta)/3}\eta_{\nu\mu}(y, \theta)$ .

Assume next that  $\hat{g}_{jk}$  satisfies the 5-dimensional Einstein equations

$$\text{Ric}_{jk}(\hat{g}) - \frac{1}{2}(\hat{g}^{pq}\text{Ric}_{pq}(\hat{g}))\hat{g}_{jk} = 0 \quad \text{in } \mathbb{R}^4 \times S^1$$

and write  $g = g_{\mu\nu}(y, \theta)$ ,  $A = A_\mu(y, \theta)$  and  $\sigma = \sigma_{\mu\nu}(y, \theta)$  in terms of Fourier series,

$$g_{\mu\nu} = \sum_{m=-\infty}^{\infty} g_{\mu\nu}^m(y)e^{im\theta}, \quad A_\mu = \sum_{m=-\infty}^{\infty} A_\mu^m(y)e^{im\theta}, \quad \sigma = \sum_{m=-\infty}^{\infty} \sigma^m(y)e^{im\theta}.$$

Here, the functions  $A_\mu^m(y)$  and  $\sigma^m(y)$  correspond to some physical fields. When  $\varepsilon$  is small, the manifold  $N$  is almost collapsed in the  $S^1$  direction and

all these functions with  $m \neq 0$  corresponds to physical fields (or particles) of a very high energy which do not appear in physical observations with a realistic energy. Thus one considers only the terms  $m = 0$  and after suitable approximations (see [54] and [65, App. E]) one observes that the matrix  $g_{\mu\nu}^0(y)$ , considered as a Lorentzian metric on  $M = \mathbb{R}^4$ , the 1-form  $A^0(y) = A_\mu^0(y)dy^\mu$ , and the scalar function  $\phi(y) = \frac{1}{\sqrt{3}}\sigma^0(y)$  satisfy

$$\text{Ric}_{\mu\nu}(g^0) - \frac{1}{2}((g^0)^{pq}\text{Ric}_{pq}(g^0))g_{\mu\nu}^0 = T_{\mu\nu} \quad \text{in } \mathbb{R}^4, \quad (206)$$

$$T_{\mu\nu} = \kappa^2 e^{\sqrt{3}\phi} \left( (g^0)^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} (g^0)^{\alpha\beta} (g^0)^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} g_{\mu\nu}^0 \right) \\ + \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} ((g^0)^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi) g_{\mu\nu}, \quad (207)$$

$$d(*F) = 0 \quad \text{in } \mathbb{R}^4, \quad (208)$$

$$\square_{g^0} \phi = \frac{1}{4} \kappa^2 e^{\sqrt{3}\phi} (g^0)^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} \quad \text{in } \mathbb{R}^4, \quad (209)$$

where  $T$  is called the stress energy tensor,  $*$  is the Hodge operator with respect to the metric  $g_{\mu\nu}^0(y)$ ,  $\nabla_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$  denotes the partial derivative,  $\square_{g^0}$  is the Laplacian with respect to the Lorentzian metric  $g_{\mu\nu}^0(y)$  (i.e. the wave operator), and  $F_{\mu\nu}(y)$  is the exterior derivative of the 1-form  $A^0(y) = A_\mu^0(y)dy^\mu$ ,  $F = dA^0$ . Physically, if we write  $F = E(y) \wedge dy^0 + B(y)$ , then  $E$  corresponds to electric field and  $B$  the magnetic flux (see [22]).

Above (208) for  $F = dA^0$  are the 4-dimensional formulation Maxwell's equations, (209) is a scalar wave equation corresponding a mass-less scalar field that interacts with the  $A^0$  field, (206)-(207) are the 4-dimensional Einstein equations in a curved space-time with stress-energy tensor  $T$ , which corresponds to the stress-energy of the electromagnetic field  $F$  and scalar field  $\phi$ . Thus Kaluza-Klein theory unified the 4-dimensional Einstein equation and Maxwell's equations. However, as the scalar wave equation did not correspond to particles observed in physical experiments the theory was forgotten for a long time due to the dawn of quantum mechanics. Later, on 1960-1980's it was re-invented in the creation of string theories when the manifold  $S^1$  was replaced by a higher dimensional manifolds, see [3]. However, the Kaluza-Klein theory is still considered as an interesting simple model close to string theory suitable for testing ideas.

To consider the relation of the Kaluza-Klein model to the main text in the paper, let us consider 5-dimensional Einstein equations with some matter

model on manifolds  $N_\varepsilon = \mathbb{R} \times M_\varepsilon$ ,  $\varepsilon > 0$  where  $(\{t\} \times M_\varepsilon, \bar{g}_\varepsilon(\cdot, t))$ ,  $t \in \mathbb{R}$  are compact 4-dimensional Riemannian manifolds. Let  $\widehat{g}_\varepsilon = -dt^2 + \bar{g}_\varepsilon(\cdot, t)$  be the background metric on  $N_\varepsilon$  and assume that the metric  $\bar{g}_\varepsilon(\cdot, t)$  is independent of the variable  $t$ . Moreover, assume that we can make do small perturbations to the matter fields in the domain  $\mathbb{R}_+ \times M_\varepsilon$  that cause the metric to become a small perturbation  $\widehat{g}_\varepsilon(\cdot, t; h) = -dt^2 + g_\varepsilon(\cdot, t; h)$  of the metric  $g_\varepsilon(\cdot, t)$ , where  $h > 0$  is a small parameter related to the amplitude of the perturbation.

By representing tensors  $g_\varepsilon(t, x; h)$  for all  $h$  at appropriate coordinates (the so called wave gauge coordinates), one obtains that the tensor  $\widetilde{g}(t, x) = \partial_h \widehat{g}_\varepsilon(t, x; h)|_{h=0}$  satisfies the linearized Einstein equations, that is, a wave equation

$$\begin{aligned} \widehat{\square} \widetilde{g}_{jk}(t, x) + b_{jk}^{lpq}(t, x) \widehat{\nabla}_l \widetilde{g}_{pq}(t, x) + c_{jk}^{pq}(t, x) \widetilde{g}_{pq}(t, x) &= \widetilde{T}_{jk}(t, x) \quad \text{on } M_\varepsilon \times \mathbb{R}, \\ \widetilde{g}_{pq}(t, x) &= 0 \quad \text{for } t < t_- \text{ for some } t_- \in \mathbb{R}, \end{aligned} \quad (210)$$

see [18, Ch. 6], where  $\widehat{\square} = \square^{\widehat{g}}$  is the wave operator,  $\widehat{\nabla} = \nabla^{\widehat{g}}$  is the covariant derivative with respect to the metric  $\widehat{g}$ , and  $\widetilde{T}$  is a source term corresponding to the perturbation of the stress-energy tensor. We mention that in realistic physical models  $\widetilde{T}$  should satisfy a conservation law but we do not discuss this issue here.

Let us now consider scalar equation analogous to (210) for a real value function  $U_\varepsilon(t, x)$  on  $\mathbb{R} \times M_\varepsilon$ ,

$$\begin{aligned} \widehat{\square} U_\varepsilon(t, x) + B_\varepsilon^\nu(x) \widehat{\nabla}_\nu U_\varepsilon(t, x) + C_\varepsilon(x) U_\varepsilon(t, x) &= F_\varepsilon(t, x) \quad \text{on } \mathbb{R} \times M_\varepsilon, \\ U_\varepsilon(t, x) &= 0 \quad \text{for } t < 0. \end{aligned}$$

We will apply to the solution  $U_\varepsilon(t, x)$  the wave-to-heat transformation

$$(\mathcal{T}f)(t) = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} e^{-\xi^2 t + i t' \xi} f(t') dt' d\xi$$

in the time variable and denote  $u_\varepsilon(t, x) := (\mathcal{T}U_\varepsilon)(t, x)$  and  $f_\varepsilon(t, x) := (\mathcal{T}F_\varepsilon)(t, x)$ . Then  $u_\varepsilon$  satisfies the heat equation

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\bar{g}_\varepsilon} + B_\varepsilon^\nu(x) \widehat{\nabla}_\nu + C_\varepsilon(x)\right) u(t, x) &= f_\varepsilon(t, x) \quad \text{on } M_\varepsilon \times \mathbb{R}_+, \\ u_\varepsilon|_{t=0} &= 0, \end{aligned}$$



where  $\Delta_{\bar{g}_\varepsilon}$  is the 3-dimensional Laplace-Beltrami operator on  $(M_\varepsilon, \bar{g}_\varepsilon)$ . Then, if we can control the source term  $F_\varepsilon$  and measure the field  $U_\varepsilon$  for the wave equation (with a measurement error) we can also produce many sources  $f_\varepsilon$  for the heat equation and compute the corresponding fields  $u_\varepsilon$ . In this paper we have assumed that we are given the values of the heat kernel, corresponding to measurements with point sources, at the  $\delta$ -dense points in the subset  $(\delta, \delta^{-1}) \times M_\varepsilon$  of the space time with some error. Due to the above relation of the heat equation to the wave equation and the hyperbolic nature of the linearized Einstein equation, the inverse problem for the point heat data can be considered as a (very much) simplified version of the question if the observations of the small perturbations of physical fields in the subset  $\mathbb{R} \times \Omega$  of an almost stationary, almost collapsed universe  $\mathbb{R} \times M_\varepsilon$  can be used to find the metric of  $\mathbb{R} \times M_\varepsilon$  in a stable way. As  $M_\varepsilon$  can be considered as a  $S^1$ -fiberbundle  $\pi : M_\varepsilon \rightarrow M_0$  on a 3-manifold  $M_0$ , it is interesting to ask if the measurements at the  $\delta$ -dense subset, where  $\delta$  is much larger than  $\varepsilon$ , can be used to determine e.g. the relative volume of the almost collapsed fibers  $\pi^{-1}(y)$ ,  $y \in M_0$ . Physically, this means the question if the macroscopic measurements be used to find information on the possible changes of the parameters of the almost collapsed structures of the universe in the space-time. We emphasize that in the questions discussed in this appendix are not related to the practical testing of string theory, but more to the philosophical question, can the properties of the almost collapsed structures in principle be observed using macroscopic observations, or not. In terms of Example 1 in Introduction of this paper, the above questions are close to the following question:

**A generalization of Example 1.** Assume that we have a manifold  $M = N \times S$ . Denote points  $x \in M$  by  $x = (y, z)$ ,  $y \in N$ ,  $z \in S$ . Assume that  $h^\sigma$  is a family of metric tensors, depending on parameter  $\sigma > 0$ , of the form  $G_{jk}(y)dy^jdy^k + \sigma^2 p_{im}(y, z)dz^i dz^m$ , where  $\sigma > 0$  is small (in physical applications,  $\sigma$  could correspond to the Planck constant). When we make measurements with accuracy  $\delta \gg \sigma$  at a  $\delta$ -dense set of measurement points, we can try to approximately find the metric  $G_{jk}(y)$  and some functions associated with  $p_{im}(y, z)$  (e.g. the dependence of the volume of  $(S, p_{im}(y, z)dz^i dz^m)$  on  $y$ ). Roughly speaking, these questions are:

(A) Can we do macroscopic reconstructions when directly unobservable  $S$ -structure is included in the physical model?

(B) Can we find parameters of  $S$ -structure via macroscopic measurements?

## 11 Appendix B: Remark on the smoothness in Calabi-Hartman [12] and Montgomery-Zippin [47]

Let  $N$  be either the manifold  $M$  or  $G \times M$ , where  $G$  is a Lie group of isometries acting on  $M$ . We consider Zygmund classes  $C_*^s(N)$ ,  $s > 0$ . To define these spaces, we cover  $N$  by a finite number of coordinate charts,  $(U_a, \Phi_a)$  with, e.g.  $\Phi_a(U_a) = Q_{2r}$ , where  $Q_r$  is a cube with a side  $2r$  and assume that

$$\bigcup_{a=1}^J (\Phi_a^{-1}(Q_r)) = N.$$

Then the definition of the norm in  $C_*^s(N)$  is analogous to the definition of these spaces in Euclidean space, cf. Sec. 2.7 [64].

Let  $\widehat{s} \in \mathbb{Z}_+$ ,  $\widetilde{s} \in (0, 1]$  satisfy  $s = \widehat{s} + \widetilde{s}$ . We say that  $f : N \rightarrow \mathbb{R}$  is in  $C_*^s(N)$ , if for some  $\delta < r$ ,

$$\|f\|_{C_*^s(N)} = \|f\|_{C(N)} + \sum_{a=1}^J \left[ \sup_{x \in Q_r} \sup_{|h| < \delta} \sum_{|\beta| \leq \widehat{s}} \frac{1}{|h|^{\widetilde{s}}} |(\Delta_h^2 \partial^\beta f_a)(x)| \right] < \infty, \quad (211)$$

where  $f_a = (\phi_j \cdot f) \circ \Phi_a^{-1}$  and  $\phi_j \in C_0^\infty(U_a)$  are functions for which  $\sum_{a=1}^J \phi_a(x) = 1$ . Here

$$(\Delta_h^2 f_a)(x) = f_a(x+h) + f_a(x-h) - 2f_a(x).$$

If condition (211) is satisfied, its rhs defines the norm of  $f$  in  $C_*^s(N)$ . We will also consider maps  $F : N \rightarrow M$  and denote that  $F \in C_*^s(N; M)$  if the coordinate representation of  $F$  in the local coordinates are  $C_*^s$ -smooth.

Note that even though the norm (211) depends on the used local coordinates, the smooth partition of unity  $\phi_a$ , and on  $\delta$ , the resulting norms are equivalent. Also, when  $s \notin \mathbb{Z}_+$ ,  $C_*^s(N)$  coincides with the Hölder spaces  $C_*^{\widehat{s}, \widetilde{s}}(N)$ .

By [64, Th. 2.7.2(2)], the norm involving only terms with the finite differences along the coordinate axis  $x^j$  of the partial derivatives along the same

coordinate axis, namely

$$\|f\|_{C_*^s}^{(1)} = \|f\|_{C(N)} + \sum_{a=1}^J \left[ \sup_{x \in Q_r} \sup_{0 < \rho < \delta} \sum_{j=1}^n \frac{1}{\rho^{\hat{s}}} \left| \left( \Delta_{\rho,j}^2 \frac{\partial^{\hat{s}} f_a}{(\partial x^j)^{\hat{s}}} \right) (x) \right| \right], \quad (212)$$

is equivalent to (211). Here

$$(\Delta_{\rho,j}^2 h)(x) = h(x + \rho e_j) + h(x - \rho e_j) - 2h(x).$$

Let  $(N^i, h^i)$ ,  $i = 1, 2$  be Riemannian manifolds and let  $B^i \subset N^i$  be metric balls on these manifolds. We will use estimates presented in [12] for map  $F : M^1 \rightarrow M^2$  which restriction to ball  $B_1$  defines an isometry  $F : (B^1, h^1) \rightarrow (B^2, h^2)$ . We note that the constants in these estimates depend only on the norms of  $h^i$ ,  $i = 1, 2$ , in the appropriate function classes  $C_*^s$  in some larger balls containing  $B^i$  and the radii of these balls. Thus, when dealing with the case when a Lie group  $G$  has isometric actions on manifold  $M$  and  $F = F_g : M \rightarrow M$  is the action of the group element  $g \in G$ , then in the case when  $M$  and  $G$  are compact, we obtain uniform estimates by covering the manifold and the Lie group with finite number of balls. In the case when  $M$  is compact but  $G$  is not, we observe that the estimates on finite collection of balls covering  $M$  and one ball  $\mathcal{B} \subset G$  for which  $\cup_{g \in G} (g\mathcal{B}) = G$  yield uniform estimates on the space  $G \times M$ .

Let  $(M, h)$  be a compact Riemannian manifold with a metric  $h \in C_*^s(M)$ ,  $s > 1$ , that is in suitable local coordinates the elements  $h_{jk}(x)$  of the metric tensor  $h$  are in  $C_*^s(M)$ . Let  $G$  be a Lie group of transformations acting on  $M$  as isometries, i.e. for any  $g \in G$  the action of  $g$ , denoted  $F_g : M \rightarrow M$  is an isometry. Let us denote  $F : G \times M \rightarrow M$ ;  $F(g, x) = F_g(x)$ . Next we will use local coordinates  $(x^j)_{j=1}^n = (x^1, \dots, x^n)$  of  $M$  and  $(g^\alpha)_{\alpha=1}^p = (g^1, \dots, g^p)$  of  $G$ . Below, we will use latin indexes  $i, k, l$  for coordinates on  $M$  and greek indexes  $\alpha, \beta, \gamma$  on coordinates on  $G$ . Thus e.g. for a function  $f : G \times M \rightarrow \mathbb{R}$  we often denote  $\partial_\alpha f(g, x) = \frac{\partial f}{\partial g^\alpha}(g, x)$  and  $\partial_j f(g, x) = \frac{\partial f}{\partial x^j}(g, x)$ .

Let us next assume that  $h \in C_*^s(M)$ ,  $s > 1$ . Our aim is to prove that then  $F : G \times M \rightarrow M$  is in  $C_*^s(G \times M; M)$ , that is, in local coordinates the components  $F^m(g, x)$  of  $F(g, x)$  are in  $C_*^s(G \times M)$ . Let us start from the case when  $1 < s \leq 2$ . Let  $g \in G$  be fixed for a while and denote  $F = F_g$ . Then we see from Th.(\*) (ii) that  $F \in C^2(M; M)$ . Using local coordinates of  $M$  in sufficiently small balls  $B_1$  and  $B_2$  satisfying  $F(B_1) \rightarrow B_2$ , we have by

[12, formula (5.2)],

$$\Gamma_{ij}^{(1),p}(x) \frac{\partial F^m}{\partial x^p} - \Gamma_{pq}^{(2),m}(F(x)) \frac{\partial F^p}{\partial x^i} \frac{\partial F^q}{\partial x^j} = \frac{\partial^2 F^m}{\partial x^i \partial x^j}, \quad (213)$$

where  $\Gamma_{ij}^{(1),p}(x)$  and  $\Gamma_{ij}^{(2),p}(x)$  are the Christoffel symbols of the metric  $h$  in  $B_1$  and  $B_2$ , correspondingly. As  $h \in C_*^s$  and  $F \in C^2$ , we see easily that all terms in the left side of formula (213) except maybe the term  $\Gamma_{pq}^{(2),m}(F(x))$  are in  $C_*^{s-1}$ . Next we consider this term and will show that as  $\Gamma_{pq}^{(2),m} \in C_*^{s-1}$  and  $F \in C^1$ , then their composition satisfies

$$\Gamma_{pq}^{(2),m} \circ F \in C_*^{s-1}(B_1). \quad (214)$$

To show this, observe that since  $F \in C^2$ , we have

$$\left| F\left(\frac{x+y}{2}\right) - \frac{1}{2}F(x) - \frac{1}{2}F(y) \right| \leq C|x-y|^2. \quad (215)$$

Let  $(s-1)/2 < t < s-1$ . Then  $\Gamma_{pq}^{(2),m} \in C^t(B_2)$  and for  $x, y \in B_1$  we have

$$\left| \Gamma_{pq}^{(2),m}\left(F\left(\frac{x+y}{2}\right)\right) - \Gamma_{pq}^{(2),m}\left(\frac{1}{2}F(x) + \frac{1}{2}F(y)\right) \right| \leq c|x-y|^{2t} \leq c'|x-y|^{s-1}.$$

Moreover, we see that

$$\begin{aligned} & \left| \Gamma_{pq}^{(2),m}(F(x)) + \Gamma_{pq}^{(2),m}(F(y)) - 2\Gamma_{pq}^{(2),m}\left(F\left(\frac{x+y}{2}\right)\right) \right| \leq \\ & \left| \Gamma_{pq}^{(2),m}(F(x)) + \Gamma_{pq}^{(2),m}(F(y)) - 2\Gamma_{pq}^{(2),m}\left(\frac{1}{2}F(x) + \frac{1}{2}F(y)\right) \right| + c'|x-y|^{s-1} \\ & \leq c|F(x) - F(y)|^{s-1} + c'|x-y|^{s-1} \leq C|x-y|^{s-1}. \end{aligned}$$

which proves inequality (214).

By [37], the space  $C_*^{s-1}(M)$  with  $s > 1$  is an algebra, that is, the pointwise multiplication satisfies  $C_*^{s-1}(M) \cdot C_*^{s-1}(M) \subset C_*^{s-1}(M)$ . Thus it follows from (213) that  $\frac{\partial^2 F^m}{\partial x^i \partial x^j} \in C_*^{s-1}(B_1)$ . This shows that  $F_g \in C_*^{s+1}(M; M)$  for  $g \in G$ .

Differentiating further (213) with respect to variables  $x^j$  and repeating the above considerations, we see that if  $h \in C_*^s(M)$  with  $s \in (3, 4]$  then  $F_g \in C_*^{s+1}(M, M)$ . Iterating this construction, we obtain

**Lemma 11.1** *Let  $(M, h)$ ,  $h \in C_*^s(M)$ ,  $s > 1$  be a compact Riemannian manifold. Let  $G$  be a Lie group of transformations acting on  $M$  as isometries, i.e., for any  $g \in G$ , the corresponding action on  $M$ ,  $F_g : M \rightarrow M$  is an isometry. Then, for each  $g \in G$  the map  $F_g$  is in  $C_*^{s+1}(M; M)$ , and the norm of  $F_g$  in  $C_*^{s+1}(M; M)$  is uniformly bounded with respect to  $g \in G$ .*

We turn now to [47]. The corresponding result in [47] which we need is as follows (see Th. on p. 212, sec. 5.2, [47]):

**Theorem 11.2** (Montgomery-Zippin) *Let  $M$  be a differentiable manifold of class  $C^k(M)$ ,  $k \in \mathbb{Z}_+$ . Let  $G$  be a Lie group of transformations acting on  $M$  so that,  $F_g(\cdot) \in C^k(M)$ , uniformly in  $g \in G$ . Then, in the local real-analytic coordinates in  $G$  and the  $C^k$  smooth coordinates of  $M$ , we have*

$$F(g, x) := F_g(x) \in C^k(G \times M).$$

Note that as  $G$  is a Lie group it has an analytic structure.

Our goal is to extend Theorem 11.2 to spaces  $C_*^s$ .

**Proposition 11.3** (Generalization of Montgomery-Zippin theorem)

*Let  $s > 1$  and  $(M, h)$  be a compact Riemannian manifold with the  $C_*^s$ -smooth coordinates. Let  $G$  be a Lie group of transformations acting on  $M$ , i.e. for any  $g \in G$ ,  $F_g : M \rightarrow M$  is a diffeomorphism. Define  $F : (G \times M) \rightarrow M$ ;  $F(g, x) = F_g(x)$ . Assume that  $F_g \in C_*^s(M; M)$ , uniformly in  $g \in G$ . Then, in the local real-analytic coordinates in  $G$  and the  $C_*^s$  smooth coordinates of  $M$ ,*

$$F \in C_*^s(G \times M; M).$$

**Proof.** Let  $1 < s' < s'' < s$ . Let us next show that the map

$$\mathcal{F} : G \rightarrow C_*^{s'}(M; M), \quad g \mapsto F_g(\cdot) \tag{216}$$

is continuous. Essentially, this follows from the facts that  $s' < s$ , our assumption that  $F_g \in C_*^s(M; M)$ , uniformly in  $g \in G$ , and that  $F(g, x)$  is uniformly continuous with respect to  $(g, x) \in G \times M$ . However, let us show this in detail. Let us first consider the case when  $1 < s \leq 2$  and a scalar, uniformly continuous function  $f : G \times M \rightarrow \mathbb{R}$ . Denoting  $f_g(x) = f(g, x)$ , and assuming

that  $g \mapsto f_g$  is continuous map  $G \rightarrow C_*^{s'}(M)$ , we have for  $h, \rho \in \mathbb{R}$ ,  $|h| < 1$ ,  $|\rho| \leq 1$

$$\begin{aligned} & \left| \frac{\partial(f_g - f_{g'})(x+h)}{\partial x^i} - \frac{\partial(f_g - f_{g'})(x)}{\partial x^i} \right. \\ & \quad \left. - \frac{1}{\rho} \left( (f_g - f_{g'})(x+h+\rho e_i) - (f_g - f_{g'})(x+h) \right. \right. \\ & \quad \left. \left. + (f_g - f_{g'})(x+\rho e_i) - (f_g - f_{g'})(x) \right) \right| \\ & \leq C(s'')\rho^{s''}, \end{aligned} \tag{217}$$

where  $C(s'')$  is uniform with respect to  $g, g' \in G$ .

On the other hand,

$$\frac{1}{|h|^{s'}} \left| \frac{\partial(f_g - f_{g'})(x+h)}{\partial x^i} - \frac{\partial(f_g - f_{g'})(x)}{\partial x^i} \right| \leq C(s'')|h|^{s''-s'}. \tag{218}$$

Thus, for any  $\varepsilon > 0$ , we can find  $h_0 > 0$  such that the left hand side of (218) is less than  $\varepsilon/2$  for  $|h| < h_0$ . Moreover, we can find  $\rho_0 > 0$  such that the right hand side of (217) is less than  $\varepsilon/2$  for  $|\rho| \leq \rho_0$ . As  $f : G \times M \rightarrow \mathbb{R}$  is continuous and, therefore, uniformly continuous, any  $g \in G$  has a neighborhood  $V_\varepsilon(g)$  in  $G$ , such that for  $g' \in V_\varepsilon(g)$  and  $x \in M$

$$|(f_g - f_{g'})(x)| < \frac{1}{8}\rho_0\varepsilon h_0^{s'}.$$

Combining the above considerations we see that if  $g' \in V_\varepsilon(g)$ , then

$$\|f_g - f_{g'}\|_{C_*^{s'}(M)} < \varepsilon.$$

Applying the above for the coordinate representation of  $F : G \times M \rightarrow M$ , we obtain (216) in the case when  $1 < s' < s \leq 2$ . Analyzing the higher derivatives similarly, we obtain (216) for all  $s > s' > 1$ .

Next let  $g^\alpha$ ,  $\alpha = 1, 2, \dots, p$  be real-analytic coordinates on  $G$  near the identity element  $\text{id}$  for which the coordinates of the identity element are  $0 = (0, 0, \dots, 0)$ . Let  $e_\alpha = \partial_{g_\alpha}|_{g=\text{id}}$ , and  $te_\alpha$ ,  $t \in \mathbb{R}$  denote the elements of the one-parameter subgroup in  $G$  generated by  $e_\alpha$ . Let  $x^j$ ,  $j = 1, 2, \dots, n$  be the local coordinates in an open set  $B \subset M$  and  $x_0 \in B$ . Near  $(\text{id}, x_0)$ , we represent  $F$  in these coordinates as  $F(g, x) = (F^m(g, x))_{m=1}^n$ . As noted

before, we will next use latin indexes  $i, k, l$  for coordinates on  $M$  and greek indexes  $\alpha, \beta, \gamma$  on coordinates on  $G$ . In particular, we will use this to indicate derivatives with respect to  $g^\alpha$  and  $x^j$ .

Consider now the identity given in [47, Lemma A a), p. 209],

$$F^m(\rho e_\alpha, x) - F^m(0, x) = \sum_{j=1}^n \left( \int_0^1 \partial_j F^m(t \rho e_\alpha, x) dt \right) \partial_\alpha F^j(0, x), \quad (219)$$

where  $\rho > 0$  is small enough, and as noted before,

$$\partial_j F^m(g, x) = \frac{\partial F^m}{\partial x^j}(g, x) \quad \text{and} \quad \partial_\alpha F^j(g, x) = \frac{\partial F^j}{\partial g^\alpha}(g, x).$$

Since

$$\partial_j F^m(0, x) = \delta_j^m, \quad x \in M,$$

it follows from the uniform continuity of  $\nabla_x F(g, x)$  with respect to  $(g, x)$  that the matrix

$$\left[ \int_0^1 \partial_j F^m(t \rho e_\alpha, x) dt \right]_{j,m=1}^n \quad (220)$$

is invertible for sufficiently small  $\rho$  (note that  $\rho > 0$  can be chosen uniformly with respect to  $x \in M$ ). Denoting the inverse matrix (220) by  $\phi_j^m(\rho e_\alpha, x)$ , we obtain from (219) the identity

$$\partial_\alpha F^m(0, x) = \phi_j^m(\rho e_\alpha, x) [F^j(\rho e_\alpha, x) - F^j(0, x)], \quad (221)$$

when  $\rho > 0$  is sufficiently small. In the following, we can choose  $\rho_0 > 0$  so that (221) is valid for all  $0 < \rho < \rho_0$  and  $x \in M$ . Using our assumption that  $F_g \in C_*^s(M; M)$ , uniformly in  $g \in G$ , we see that the matrix (220) is in  $C_*^{s-1}(M)$  and thus its inverse matrix satisfies  $\phi_j^m(\rho_0 e_\alpha, \cdot) \in C_*^{s-1}(M)$ . As  $F_g \in C_*^s(M; M)$  uniformly with respect to  $g \in G$ , formula (221) implies that

$$\partial_\alpha F^m(0, x) \in C_*^{s-1}(M). \quad (222)$$

Our next goal is to show that

$$\partial_\alpha F^m(g, x) = \frac{\partial F^m(g, x)}{\partial g^\alpha} \in C_*^{s-1}(G),$$

uniformly with respect to  $x \in M$ . To this end, let  $g_0 \in G$  and  $x \in M$  be fixed, and consider an element  $g \in G$  which varies in a neighborhood of  $g_0^{-1}$ . Let us consider function  $\tilde{g}(g) := g_0 g^{-1}$ ,  $\tilde{g} : G \rightarrow G$  and  $\tilde{y}(g) = F_g(x)$ ,  $\tilde{y} : G \rightarrow M$ . The group  $G$  has real-analytic local coordinates  $(g^\alpha)_{\alpha=1}^p = (g^1, \dots, g^p)$  near  $g_0$ , and as the group operation  $(g', g'') \mapsto g' \cdot (g'')^{-1}$  is real-analytic, also the local coordinates of  $\tilde{g}(g)$ , denoted  $\tilde{g}^\alpha(g^1, \dots, g^p)$  are real analytic. Then,

$$F(\tilde{g}(g), \tilde{y}(g)) = F(g_0 g^{-1}, F_g(x)) = F(g_0 g^{-1} g, x) = F(g_0, x).$$

Thus,  $F(\tilde{g}(g), \tilde{y}(g))$  is independent of  $g$ , we obtain using the chain rule

$$\begin{aligned} 0 &= \frac{\partial}{\partial g^\alpha} (F(\tilde{g}(g), \tilde{y}(g))) \\ &= \partial_\beta F^m(\tilde{g}(g), \tilde{y}(g)) \frac{\partial \tilde{g}^\beta(g)}{\partial g^\alpha} + \partial_j F^m(\tilde{g}(g), \tilde{y}(g)) \frac{\partial \tilde{y}^j(g)}{\partial g^\alpha}. \end{aligned} \quad (223)$$

Let us next use that fact that for all  $z \in M$  we have  $\partial_j F^m(g, z)|_{g=id} = \delta_j^m$ . As derivatives of the map  $(g, z) \mapsto \partial_j F^m(g, z)$  is continuous, the matrix  $(\partial_j F^m(g, z))_{j,m=1}^n$  is invertible for  $(g, z)$  near  $(id, z)$ . Let us denote this inverse matrix by  $\Phi_j^m(g, z)$ . Composing this function with  $\tilde{g}(g)$  and  $\tilde{y}(g)$  we see that the matrix  $(\partial_j F^m(\tilde{g}(g), y(g)))_{j,m=1}^n$  has an inverse  $\Phi_j^m(\tilde{g}(g), y(g))$  when  $g$  is sufficiently near to  $g_0$ . Then we obtain from (223)

$$\begin{aligned} \partial_\alpha F^k(g, x) &= \partial_\alpha \tilde{y}^m(g, x) \\ &= -\Phi_k^m(\tilde{g}(g), F(g, x)) \cdot \partial_\beta F^k(\tilde{g}(g), F(g, x)) \cdot \partial_\alpha \tilde{g}^\beta(g). \end{aligned} \quad (224)$$

Let us now evaluate (224) at  $g = g_0$ . As  $\Phi_k^m(\tilde{g}(g_0), F(g_0, x)) = \delta_k^m$ , and in the local coordinates of  $G$  we have  $\tilde{g}(g_0) = 0$ , we obtain

$$\partial_\alpha F^m(g_0, x) = -\partial_\beta F^m(0, F(g_0, x)) \cdot \partial_\alpha \tilde{g}^\beta(g_0). \quad (225)$$

Note that above  $g_0$  and  $x$  were arbitrary, even though we considered those as fixed parameters. Next we change our point of view and will consider those as variables. For this end, we use variable  $g \in G$  instead of  $g_0$  so that (225) reads as

$$\partial_\alpha F^m(g, x) = -\partial_\beta F^m(0, F(g, x)) \cdot \partial_\alpha \tilde{g}^\beta(g). \quad (226)$$

Here  $g \mapsto \partial_\alpha \tilde{g}^\beta(g)$  is real-analytic and by formula (222), the map  $z \mapsto \partial_\beta F^m(0, z)$  is in  $C_*^{s-1}(M)$ .



Let us next show that

$$\partial_\beta F^m(0, F(g, x)) \in C_*^{s-1}(G \times M). \quad (227)$$

When this is shown, the fact that  $\partial_\alpha \tilde{g}^\beta(g)$  is real-analytic and the equation (226) imply that  $\partial_\alpha F^m(g, x) \in C_*^{s-1}(G \times M)$ .

To show (227), we start by considering the case when  $1 < s < 2$ . Then, using (222) and that fact that  $F \in C^1(G \times M)$  by Theorem 11.2, we obtain in local coordinates

$$\begin{aligned} & |\partial_\beta F^m(0, F(g+h, x)) - \partial_\beta F^m(0, F(g, x))| \\ & \leq C |F(g+h, x) - F(g, x)|^{s-1} \leq C' |h|^{s-1}, \end{aligned} \quad (228)$$

where  $C, C' > 0$  are uniform with respect to  $(g, x) \in G \times M$ . Hence (227) is valid, i.e.,  $\partial_\alpha F^m(g, x) \in C_*^{s-1}(G \times M)$ . Combining this with (226) we see that for any  $x \in M$  the function  $g \mapsto \partial_\alpha F^m(g, x)$  is in  $C_*^{s-1}(G)$  and its norm in  $C_*^{s-1}(G)$  is bounded by a constant which is independent of  $x \in M$ . By assumption, functions  $x \mapsto \partial_j F^m(g, x)$  are in  $C_*^{s-1}(M)$ , uniformly in  $g \in G$ . Thus, by (212), i.e., [64, Thm. 2.7.2(2)], the map  $F : G \times M \rightarrow M$  is  $C_*^s$ -smooth with respect to both  $g$  and  $x$ , that is,

$$F(g, x) \in C_*^s(G \times M). \quad (229)$$

Next, let us consider the case when  $s = 2$ . To apply (212), we need to estimate

$$T_\beta^k = \partial_\beta F^k(0, F(g, x))$$

and in particular its finite difference

$$\Delta_h^2(T_\beta^k) = \partial_\beta F^k(0, F(g+h, x)) + \partial_\beta F^k(0, F(g-h, x)) - 2\partial_\beta F^k(0, F(g, x)).$$

Using (229) we observe that for  $h = (h^\alpha)_{\alpha=1}^p$

$$|F^k(g+h, x) - F^k(g, x) - \partial_\alpha F^k(g, x) \cdot h^\alpha| \leq C_s |h|^{s'}, \quad \text{for any } s' < 2.$$

On the other hand, by (229), we have

$$\partial_\beta F^k(0, F(g, x)) \in C^t(G \times M), \quad \text{for any } t < 1.$$

Let us combine these two observations in the case when  $s' = 3/2$  and  $t = 2/3$ . Then

$$\begin{aligned}
\Delta_h^2(T_\beta^k) &= \partial_\beta F^k(0, F(g, x) + (\partial_\alpha F^k)(g, x) h^\alpha + O(|h|^{s'})) \\
&\quad + \partial_\beta F^k(0, F(g, x) - (\partial_\alpha F^k)(g, x) h^\alpha + O(|h|^{s'})) \\
&\quad - 2\partial_\beta F^k(0, F(g, x)) \\
&= O((|h|^{s'})^t) = O(|h|).
\end{aligned} \tag{230}$$

Applying (222) with  $s = 2$  to estimate the finite differences of  $T_\beta^k$  in the  $x^j$  directions and (230) to estimate the finite differences of  $T_\beta^k$  in the  $g^\alpha$  directions in the formula (212) for the Zygmund norm, we obtain  $T_\beta^k = \partial_\beta F^k(g, x) \in C_*^1(G \times M)$ . Moreover, by our assumption, functions  $x \mapsto \partial_j F^m(g, x)$  are in  $C_*^1(M)$ , uniformly in  $g \in G$ . Thus, by applying again (212), we see that  $F(g, x) \in C_*^2(G \times M)$ .

Next we consider the case when  $2 < s \leq 3$ . By differentiating formula (226) with respect to  $g^\beta$ , we obtain

$$\begin{aligned}
\partial_\alpha \partial_\beta F^k(g, x) &= -\frac{\partial}{\partial g^\beta} \left( \partial_\gamma F^k(0, F(g, x)) \partial_\alpha \tilde{g}^\gamma(g) \right) \\
&= (\partial_\gamma \partial_j F^k(0, F(g, x)) \partial_\beta F^j(g, x) \partial_\alpha \tilde{g}^\gamma(g) \\
&\quad + \partial_\gamma F^k(0, F(g, x)) \frac{\partial^2 \tilde{g}^\gamma}{\partial g^\alpha \partial g^\beta}(g)).
\end{aligned} \tag{231}$$

We see easily that all terms in the right hand side of equation (231), except maybe the term  $\partial_\gamma \partial_j F^k(0, F(g, x))$ , are in  $C_*^1(G \times M)$ . To analyze this term, we observe that by (222) we have  $\partial_\gamma \partial_j F^k(0, y) \in C_*^{s-2}(M)$ . Thus considerations similar to those leading to the inequality (227) show that

$$\partial_\gamma \partial_j F^k(0, F(g, x)) \in C_*^{s-2}(G \times M).$$

Hence by (231),

$$\partial_\alpha \partial_\beta F^k(g, x) \in C_*^{s-2}(G), \tag{232}$$

uniformly with respect to  $x \in M$ . By assumption  $F_g(\cdot) \in C_*^s(M)$ , uniformly in  $g \in G$ , so that

$$\partial_i \partial_j F^k(g, x) \in C_*^{s-2}(M), \tag{233}$$

uniformly with respect to  $g \in G$ .

Apply the inequality (212), with (232) and (233) we see that  $F \in C_*^s(G \times M)$ .

Iterating the above procedure we obtain (227) for all  $s > 1$ . QED

Combining Lemma 11.1 and Proposition 11.3 we obtain

**Corollary 11.4** *Let  $M$  be a compact,  $C_*^{s+1}$ -smooth differentiable manifold,  $s > 1$  and  $h \in C_*^s(M)$  be a Riemannian metric on  $M$ . Let  $G$  be a group acting on  $(M, g)$  for which the actions of the elements  $g \in G$ ,  $F_g : M \rightarrow M$  are isometries. Then  $F(g, x) \in C_*^{s+1}(G \times M)$ .*

## 12 Appendix C: Operator-theoretical approach to operator $\Delta_X$ .

In this section we redefine the operator  $\Delta_X$  extending on Fukaya's [24, Sec. 7]. Recall, see Theorem 2.3 (2), that if  $(X, \mu_X) = \lim_{mGH}(M_i, \mu_i)$ , then

$$(X, \mu_X) = \pi(Y) := (Y, \mu_Y)/O(n), \quad (Y, \mu_Y) = \lim_{mGH}(TM_i, \mu_i^*), \quad (234)$$

where  $\mu_i^*$  is the natural Riemannian probability measure on  $TM_i$  inherited from  $(M_i, h_i)$  and  $\pi : Y \rightarrow X$  is a Riemannian submersion. Recall also that,

$$h_Y \in C_*^2(Y), \quad \rho_Y, \rho_Y^{-1} \in C_*^2(Y), \quad (235)$$

and  $\mu_X, \rho_X$  are related to  $\mu_Y, \rho_Y$  through (40), (41). since  $O(n)$  acts as a subgroup of isometries on  $Y$ , by Corollary 11.4,

$$\begin{aligned} F_O : O(n) \times Y &\rightarrow Y, \quad F_O(o, y) = o(y), \quad o \in O(n), \quad y \in Y, \\ F_O &\in C_*^3(O(n) \times Y). \end{aligned} \quad (236)$$

Denote by  $L_O^2(Y, \mu_Y)$  the subspace of  $L^2(Y)$  of  $O(n)$ -invariant functions and by  $L_\perp^2(Y)$  its orthogonal complement,

$$L^2(Y, \mu_Y) = L_O^2(Y) \oplus L_\perp^2(Y).$$

Then  $L_O^2(Y)$ ,  $L_\perp^2(Y)$  are invariant subspaces of the operator  $\Delta_Y$ , where, by (235),  $\mathcal{D}(\Delta_Y) = W^{2,2}(Y)$ . Indeed, if  $u^* \in W_O^{2,2}(Y)$ , then  $\Delta_Y u^* \in L_O^2(Y)$  and, if  $u^* \in W^{2,2}(Y) \cap L_\perp^2(Y)$ , then  $\Delta_Y u^* \in L_\perp^2(Y)$ .

It follows from Lemma 4.1 that  $L_O^2(Y)$ ,  $L_\perp^2(Y)$  are invariant subspaces of  $\Delta_Y$  so that

$$\Delta_Y = \Delta_O \oplus \Delta_\perp, \quad (237)$$

where  $\Delta_O$ ,  $\Delta_\perp$  are the parts of  $\Delta_Y$  in  $L_O^2(Y)$ ,  $L_\perp^2(Y)$ , correspondingly. In the future, we denote by  $\{\lambda_j^O, \phi_j^O\}$  and  $\{\lambda_j^\perp, \phi_j^\perp\}$  the eigenpairs of  $\Delta_O$ ,  $\Delta_\perp$ , correspondingly.

Next we use the fact that, due to (40),

$$\pi_* : L_O^2(Y, \mu_Y) \rightarrow L^2(X, \mu_X) \quad (238)$$

is an isometry. Thus, we can define a self-adjoint operator  $A$  in  $L^2(X)$  by

$$\begin{aligned} Au &= \pi_* \circ \Delta_Y \circ \pi^* u, \\ \mathcal{D}(A) &= \pi_* (\mathcal{D}(\Delta_O)) = \pi_* (W_O^{2,2}(Y)). \end{aligned} \quad (239)$$

On the other hand, modifying [24, Sec. 7], we define the Dirichlet form

$$\begin{aligned} a_O[u^*] &= \int_Y |du^*(y)|_{h_Y}^2 d\mu_Y = \int_{\pi^{-1}(X^{reg})} |du^*(y)|^2 d\mu_Y, \\ \mathcal{D}(a_O) &= C_O^{0,1}(Y), \end{aligned}$$

where we use the fact that  $du^* \in L^\infty(Y)$  if  $u^* \in C^{0,1}(Y)$  and  $\mu_Y(\pi^{-1}(X^{reg})) = 1$ . Using the Kato's theory of quadratic forms, [43],  $a_O$  is closable with  $\mathcal{D}(\overline{a_O}) = W_O^{1,2}(Y)$  and the associated self-adjoint operator is  $\Delta_O$ . Observe that

$$\pi_* : C_O^{0,1}(Y) \rightarrow C^{0,1}(X) \quad (240)$$

is an isometry. To see this just recall that

$$d_X(x, x') = d_Y(\pi^{-1}(x), \pi^{-1}(x')), \quad d_X(x, x') = d_Y(y, y'), \quad (241)$$

for some  $y \in \pi^{-1}(x)$ ,  $y' \in \pi^{-1}(x')$ . Let, for  $u \in C^{0,1}(X)$ ,

$$\begin{aligned} a_X[u] &= \int_{X^{reg}} |du|_{h_X}^2 d\mu_X = \int_{\pi^{-1}(X^{reg})} |d(\pi^* u)|_{h_Y}^2 d\mu_Y \\ &= \int_Y |d(\pi^* u)|_{h_Y}^2 d\mu_Y = a_O[\pi^* u], \end{aligned}$$

where the middle inequality follows from (235), see also Lemma 7.1 of [24]. Then  $a_X$  is closable with

$$\mathcal{D}(\overline{a_X}) = \pi_*(W^{1,2}(Y) \cap L_O^2(Y)).$$

This defines an associated self-adjoint operator,  $A'$  in  $L^2(X)$ . Using the distribution duality, we can see that, in local coordinates on  $X^{reg}$ , this operator is given by

$$A'u(x) = -\frac{1}{\sqrt{h_X}\rho_X}\partial_i\left(\sqrt{h_X}h_X^{ij}\rho_X\partial_j u(x)\right).$$

This implies that  $\mathcal{D}(A') \subset W^{2,2}(X^{reg}, \mu_X)$  and it is natural to use the notation  $\Delta_X$  for  $A'$ .

Moreover, since the operator, associated with  $\overline{a_O}$ , is  $\Delta_O$ , the operator  $\Delta_X$  is unitary equivalent to  $\Delta_O$  with equation (239) being valid for  $\Delta_X$ , so that

$$\begin{aligned} A' &= \Delta_X = \pi_* \circ \Delta_Y \circ \pi^* u = A, \\ \mathcal{D}(\Delta_X) &= \pi_*(W_O^{2,2}(Y)) \subset W^{2,2}(X^{reg}). \end{aligned} \tag{242}$$

In particular,

$$\text{spec}(\Delta_X) = \text{spec}(\Delta_O) \subset \text{spec}(\Delta_Y),$$

which, due to the estimate  $|R(Y)| \leq \frac{\text{Lambd}_F^2}{\overline{\mathfrak{M}}_p(n, \Lambda, D)}$ ,  $\text{diam}(Y) \leq D$ , provides another proof of Proposition 5.2 for  $(X, p, \mu_X) \in \overline{\mathfrak{M}}_p(n, \Lambda, D)$ . Using again (242), we see that

$$\phi_j = \pi_*(\phi_j^O), \tag{243}$$

are the orthonormal eigenfunctions on  $X$  corresponding to  $\lambda_j = \lambda_j^O$ .

**Remark 12.1** Note that, if  $M_i \rightarrow \{p\}$ , i.e. collapse to a point, then  $O(n)$  acts transitively on the corresponding  $Y$ . Thus,  $L_O^2(Y)$  is 1-dimensional and consists only of constant functions. Therefore,  $\text{spec}(\Delta_O) = 0$ . Similarly,  $\Delta_X$  is the operator of multiplication by 0 in the space of  $L^2$ -functions on  $\{p\}$ , i.e. constants. The results of [24] remain valid for this case also.

## References

- [1] Anderson M. T., Convergence and rigidity of manifolds under Ricci curvature bounds. *Invent. Math.*, **102** (1990), no. 2, 429-445.
- [2] Anderson M., Katsuda A., Kurylev Y., Lassas M., Taylor M. Boundary regularity for the Ricci equation, geometric convergence, and Gelfand's inverse boundary problem. *Invent. Math.* **158** (2004), 261-321
- [3] Appelquist T.; Chodos A.; Freund P. Modern Kaluza-Klein Theories, Frontiers in Physics, Addison Wesley 1987, 619 pp.
- [4] Astala K.; Päiväranta L. Calderón's inverse conductivity problem in the plane. *Annals of Math.*, **163** (2006), 265-299.
- [5] Belishev, M. I., An approach to multidimensional inverse problems for the wave equation. (Russian) *Dokl. Akad. Nauk SSSR*, **297** (1987), no. 3, 524-527.
- [6] Bemelmans, J.; Min-Oo; Ruh, E. A., Smoothing Riemannian metrics, *Math. Z.*, **188** (1984), no. 1, 69 – 74.
- [7] Berard, P.; Besson, G.; Gallot, S. Embedding Riemannian manifolds by their heat kernel. *Geom. Funct. Anal.*, **4** (1994), no. 4, 373-398.
- [8] Bergh J., Löfström J. Interpolation spaces. An introduction. Grundlehren der Mathem Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976. x+207 pp,
- [9] Billingsley, P., Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.
- [10] Burago D., Burago Y.D., Ivanov S. A Course in Metric Geometry. AMS, Providence RI, Graduate Studies in Maths, **33**, 2001, 421 pp.
- [11] Burago, Yu.; Gromov, M.; Perelman, G., A. D. Aleksandrov spaces with curvatures bounded below. (Russian) *Uspekhi Mat. Nauk* **47** (1992), no. 2(284), 3-51, 222; translation in Russian Math. Surveys **47** (1992), no. 2, 1-58

- [12] Calabi E, Hartman P. On the smoothness of isometries. *Duke Math. J.*, bf 37 (1970), 741-750.
- [13] Calderón, A.-P. On an inverse boundary value problem. *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pp. 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980.
- [14] Cheeger, J., Finiteness theorems for Riemannian manifolds. *Amer. J. Math.* **92** (1970), 61 – 74.
- [15] Cheeger J., Fukaya K., Gromov, M. Nilpotent structures and invariant metrics on collapsed manifolds. *J. Amer. Math. Soc.* **5** (1992), 327-372.
- [16] Cheng, S. Y., Eigenfunctions and nodal sets, *Comment. Math. Helv.* **51** (1976), 43–55.
- [17] Chiang Y.-J. Spectral geometry of V-manifolds and its application to harmonic maps, in: *Pros. Symp. Pure Math.*, **54**, Part 1, 93-99, AMS (1990)
- [18] Choquet-Bruhat, Y. General relativity and the Einstein equations. *Oxford Mathematical Monographs*. Oxford University Press, Oxford, 2009. xxvi+785 pp.
- [19] Courant R. and Hilbert D. *Methods of Mathematical Physics*.
- [20] Dryden E.B., Gordon C.S., Greenwald S.J., Webb D.L. Asymptotic expansion of the heat kernel for orbifolds. *Michigan Math. J.* **56** (2008), 205-238.
- [21] Edmunds D.E., Evans W.D. *Spectral Theory and Differential Operators*. Oxford Univ. Press, 1987, 574pp.
- [22] Frankel, Th. *The geometry of physics. An introduction*. Cambridge University Press, Cambridge, 1997. xxii+654 pp.
- [23] Fukaya, K. Collapsing Riemannian manifold to ones of lower dimension *J. Differential Geom.* **25** (1987), 139–156.
- [24] Fukaya, K.: Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. *Invent. Math.* **87** (1987), no. 3, 517–547.

- [25] Fukaya, K. A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. *J. Differential Geom.* **28** (1988), no. 1, 1–21.
- [26] Fukaya, K., Hausdorff convergence of Riemannian manifolds and its applications, *Differential and analytic geometry*, Adv. Stud. Pure Math., **18**, Academic Press, Boston, MA, (1990), 143–238.
- [27] Fukaya, K., Collapsing Riemannian manifolds to ones with lower dimension, *J. Math. Soc. Japan*, **41**, (1989), 333–356.
- [28] Fukaya, K.; Yamaguchi, T., The fundamental groups of almost nonnegatively curved manifolds, *Ann. of Math.*, **136**, (1992), 253 – 333.
- [29] Gel’fand, I. M. Some aspects of functional analysis and algebra. 1957 Proc. Intern.Cong. of Mathems, Amsterdam, 1954, Vol. 1 pp. 253-276 ; North-Holland Publishing Co., Amsterdam
- [30] Gilbarg, D.; Trudinger N. Elliptic Partial Differential Equations of Second Order, *Grundlehr. der Mathem. Wissensch.*, **224**, Springer, Berlin-New York, (1983), 513pp.
- [31] Greene, R. E.; Wu, H., Lipschitz convergence of Riemannian manifolds, *Pacific J. Math.*, **131** (1988), 119 – 141.
- [32] Greenleaf A.; Lassas M.; Uhlmann G., On nonuniqueness for Calderón’s inverse problem, *Math. Res. Lett.* **10** (2003), no. 5-6, 685-693.
- [33] Greenleaf A.; Kurylev Y.; Lassas M.; Uhlmann G.: Full-wave invisibility of active devices at all frequencies. *Comm. Math. Phys.* **275** (2007), 749-789.
- [34] Greenleaf, A.; Kurylev, Y.; Lassas, M.; Uhlmann, G. Invisibility and inverse problems. *Bull. Amer. Math. Soc. (N.S.)* **46** (2009), no. 1, 55–97.
- [35] Greenleaf, A.; Kurylev, Y.; Lassas, M.; Uhlmann, G. Cloaking devices, electromagnetic wormholes, and transformation optics. *SIAM Rev.* **51** (2009), no. 1, 3–33
- [36] Gromov, M. : Structures métriques pour les variétés riemanniennes, Edited by J. Lafontaine and P. Pansu, Textes Mathématiques [Mathematical Texts] **1**, CEDIC, Paris, 1981.



- [37] Hörmander, L. Lectures on nonlinear hyperbolic differential equations. Mathém. and Applic. (Berlin), 26. Springer-Verlag, Berlin, 1997. viii+289 pp.
- [38] Kasue, A., Measured Hausdorff convergence of Riemannian manifolds and Laplace operators. II. Complex geometry (Osaka, 1990), 97 – 111, Lecture Notes in Pure and Appl. Math., **143**, Dekker, New York, 1993.
- [39] Kenig, C. E.; Sjöstrand, J.; Uhlmann, G. The Calderon problem with partial data. Ann. of Math. (2) **165** (2007), no. 2, 567–591.
- [40] Katchalov, A., Kurylev, Y. Multidimensional inverse problem with incomplete boundary spectral data. Comm. PDE **23** (1998), no. 1-2, 27–59.
- [41] Katchalov, A., Kurylev, Y., Lassas, M. Inverse Boundary Spectral Problems, Chapman Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 123. (2001), 290pp.
- [42] Katchalov, A.; Kurylev, Y.; Lassas, M.; Mandache, N. Equivalence of time-domain inverse problems and boundary spectral problems. Inverse Problems **20** (2004), no. 2, 419-436.
- [43] Kato T. Perturbation Theory for Linear Operators. Springer, Berlin-New York (1976).
- [44] Krupchyk, K., Kurylev, Y., Lassas, M.: Inverse spectral problems on a closed manifold. Journal de Mathematique Pures et Appliquees **90** (2008), 42-59.
- [45] Krylov N. V. Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, Grad. Studies in Maths, v. 96, AMS, 2008, 357pp
- [46] Kurylev, Y., Lassas, M. Hyperbolic inverse problem with data on a part of the boundary. *Differential Equations and Mathematical Physics (Birmingham, AL, 1999)*, 259–272, AMS/IP Stud. Adv. Math., **16**, Amer. Math. Soc., 2000.
- [47] Montgomery D., Zippin L. Topological Transformation Groups. R.E.Kruger Publ., N.Y, 1974. 289pp.

- [48] Nachman, A. Reconstructions from boundary measurements. *Ann. of Math.* (2) **128** (1988), no. 3, 531–576.
- [49] Nachman, A. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.* (2) **143** (1996), no. 1, 71–96.
- [50] Nachman, A., Sylvester, J. Uhlmann, G. An  $n$ -dimensional Borg-Levinson theorem. *Comm. Math. Phys.* **115** (1988), no. 4, 595–605.
- [51] O’Neill, G. Semi-Riemannian Geometry with Applications to Relativity. Acad. Press, New York-Boston-London, 1983, 468 pp.
- [52] Peters, S., Convergence of Riemannian manifolds, *Compositio Math.* **62** (1987), no. 1, 3–16.
- [53] Petrunin, A. Parallel transportation for Alexandrov space with curvature bounded below. *Geom. Funct. Anal.* **8** (1998), no. 1, 123–148.
- [54] Strominger, A., Kaluza-Klein compactifications, supersymmetry, and Calabi-Yau Spaces; in the collection Quantum fields and strings: A course for mathematicians, Volumes 1 and 2, by Deligne P. et al (eds.), AMS, Providence, RI, 1999, vol. 1 and 2, 723 pp.
- [55] Satake I. The Gauss-Bonnet theorem for V-manifolds. *J. Math. Soc. Japan*, **9** (1957), no. 4, 464–492.
- [56] Shefel S. Z., Smoothness of a conformal map of Riemannian spaces, *Sibirsk. Mat. Zh.*, **23** (1982), 153–159.
- [57] Shiohama, K., An introduction to the geometry of Alexandrov spaces, Lecture Notes Series, **8**, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993.
- [58] Stein E. Topics in Harmonic Analysis Related to the Littlewood-Paley Theorem, *Ann. Math. Studies*, N.63 (1970), 146pp, Princeton Univ. Press.
- [59] Sylvester, J., Uhlmann, G. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.* (2) **125** (1987), no. 1, 153–169.

- [60] Tataru D., Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations* 20 (1995), no. 5-6, 855–884.
- [61] Taylor, M. E. *Partial differential equations III. Nonlinear equations*. Second edition. *Appl. Math. Sciences*, 117. Springer, New York, 2011. xxii+715 pp
- [62] Taylor M. Existence and regularity of isometries. *Trans. of AMS*, 358 (2006), no. 6, 2415-2423.
- [63] Thurston, W.P. *The Geometry and Topology of Three-Manifolds*, <http://www.msri.org/publications/books/gt3m/>
- [64] Triebel H. *Interpolation Spaces. Function Theory. Differential Operators*. Second edition. Johann Ambrosius Barth, Heidelberg, 1995. 532 pp
- [65] Wald, R. *General relativity*, University of Chicago press, 1984
- [66] Zworski, M. Singular part of the scattering matrix determines the obstacle. *Osaka J. Math.* **38** (2001), no. 1, 13–20